

Strichartz estimates for Schrödinger equations with variable coefficients and unbounded potentials. II. Superquadratic potentials

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Abstract

We prove local-in-time Strichartz estimates with loss of derivatives for Schrödinger equations with variable coefficients and potentials, under the conditions that the Hamilton flow, generated by the kinetic energy, is nontrapping and convex and that the electric (resp. magnetic) potential can blow up superquadratically (resp. superlinearly) at spatial infinity in both of positive and negative directions. This is a generalization and improvement of the result by Yajima-Zhang [38].

1 Introduction

Let \tilde{P} be a Schrödinger operator on \mathbb{R}^d with variable coefficients $g^{jk}(x)$ and electromagnetic potentials $V(x)$ and $A(x) = (A_1(x), \dots, A_d(x))$ of the following form:

$$\tilde{P} = \frac{1}{2}(D_j - A_j(x))g^{jk}(x)(D_k - A_k(x)) + V(x), \quad D_j := -i\partial/\partial x_j, \quad x \in \mathbb{R}^d.$$

Here and, in the sequel, we use the Einstein summation convention.

Assumption A. $g^{jk}, A_j, V \in C^\infty(\mathbb{R}^d; \mathbb{R})$, and $(g^{jk}(x))_{j,k}$ is symmetric and uniformly elliptic in the sense that $g^{jk}(x)\xi_j\xi_k \geq c|\xi|^2$ on \mathbb{R}^{2d} with some positive constant $c > 0$. Moreover, there exists $m \geq 2$ such that, for any $\alpha \in \mathbb{Z}_+^d := \mathbb{N}^d \cup \{0\}$,

$$|\partial_x^\alpha g^{jk}(x)| + \langle x \rangle^{-m/2} |\partial_x^\alpha A_j(x)| + \langle x \rangle^{-m} |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|}. \quad (1.1)$$

Here $\langle x \rangle$ stands for $\sqrt{1 + |x|^2}$. Under Assumption A, the operator \tilde{P} , with the domain $C_0^\infty(\mathbb{R}^d)$, is symmetric in $L^2(\mathbb{R}^d)$. Let P be any one of its self-adjoint extensions. Then we consider the time-dependent Schrödinger equation

$$i\partial_t u = Pu, \quad t \in \mathbb{R}; \quad u|_{t=0} = u_0 \in L^2(\mathbb{R}^d). \quad (1.2)$$

The solution is given by $u(t) = e^{-itP}u_0$ by Stone's theorem, where e^{-itP} denotes a unique unitary propagator on $L^2(\mathbb{R}^d)$ generated by P .

In this paper we are interested in the (local-in-time) *Strichartz estimates* of the forms:

$$\|e^{-itP}u_0\|_{L_T^p L^q} \leq C_T \|\langle H \rangle^\gamma u_0\|_{L^2}, \quad (1.3)$$

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where $\gamma \geq 0$, $L_T^p L^q := L^p([-T, T]; L^q(\mathbb{R}^d))$ and (p, q) satisfies the following *admissible condition*

$$2 \leq p, q \leq \infty, \quad 2/p = d(1/2 + 1/q), \quad (d, p, q) \neq (2, 2, \infty). \quad (1.4)$$

Strichartz estimates can be regarded as L^p -type smoothing properties of Schrödinger equations and have been widely used in the study of nonlinear Schrödinger equations (see, *e.g.*, [7]).

When $g^{jk} = \delta_{jk}$ and $A \equiv 0$, the following has been proved by Yajima-Zhang [38]:

Theorem 1.1 (Theorem 1.3 of [38]). *Let $d \geq 1$ and $H = -\Delta/2 + V$ satisfy Assumption A and*

$$V(x) \geq C\langle x \rangle^m \quad \text{for } |x| \geq R, \quad (1.5)$$

with some $R, C > 0$. Then, for any $\varepsilon, T > 0$ and (p, q) with (1.4) there exists $C_{T,\varepsilon} > 0$ such that

$$\|e^{-itH} u_0\|_{L_T^p L^q} \leq C_{T,\varepsilon} \|\langle H \rangle^{\frac{1}{p}(\frac{1}{2} - \frac{1}{m}) + \varepsilon} u_0\|_{L^2}. \quad (1.6)$$

In this paper we extend this theorem to the variable coefficient case under some geometric conditions on the Hamilton flow generated by the kinetic energy. Furthermore, we remove the additional loss $\langle H \rangle^\varepsilon$ for the flat case.

To state our main results, we introduce some notation on the classical system. Let $k(x, \xi)$ be the classical kinetic energy function:

$$k(x, \xi) = \frac{1}{2} g^{jl}(x) \xi_j \xi_l, \quad x, \xi \in \mathbb{R}^d.$$

We denote by $(y_0(t, x, \xi), \eta_0(t, x, \xi))$ the Hamilton equation generated by k , *i.e.*, the solution to

$$\frac{d}{dt} y_0(t) = \frac{\partial k}{\partial \xi}(y_0(t), \eta_0(t)), \quad \frac{d}{dt} \eta_0(t) = -\frac{\partial k}{\partial x}(y_0(t), \eta_0(t))$$

with the initial condition $(y_0(0, x, \xi), \eta_0(0, x, \xi)) = (x, \xi)$. Note that the Hamiltonian vector field $H_k = \partial_\xi k \cdot \partial_x - \partial_x k \cdot \partial_\xi$ is complete on \mathbb{R}^{2d} since $(g^{jk})_{j,k}$ is uniformly elliptic. $(y_0(t), \eta_0(t))$ thus exists for all $t \in \mathbb{R}$. To control its asymptotic behavior, we then impose the following conditions:

Assumption B. (1) (Nontrapping condition) For any $(x, \xi) \in \mathbb{R}^{2d}$ with $\xi \neq 0$,

$$|y_0(t, x, \xi)| \rightarrow +\infty \quad \text{as } t \rightarrow \pm\infty.$$

(2) (Convexity near infinity) There exists $f \in C^\infty(\mathbb{R}^d)$ satisfying $f \geq 1$ and $\lim_{|x| \rightarrow +\infty} f(x) = +\infty$ such that $\partial_x^\alpha f \in L^\infty(\mathbb{R}^d)$ for any $|\alpha| \geq 2$ and that, for some constants $c, R > 0$,

$$H_k(H_k f)(x, \xi) \geq c k(x, \xi)$$

for all $x, \xi \in \mathbb{R}^d$ with $f(x) \geq R$.

Remark 1.2. It is easy to see that if the quantity

$$\sup_{|\alpha| \leq 2} \langle x \rangle^{|\alpha|} |\partial_x^\alpha (g^{jk}(x) - \delta_{jk})|$$

is sufficiently small, then $\partial_t^2(|y_0(t)|^2) \gtrsim |\xi|^2$ and hence Assumption B (1) holds. Under the same condition, Assumption B (2) also holds with $f(x) = 1 + |x|^2$. Moreover, if $g^{jk}(x) = (1 + a_1 \sin(a_2 \log r)) \delta_{jk}$ for $a_1 \in \mathbb{R}, a_2 > 0$ with $a_1^2(1 + a_2^2) < 1$ and for $r = |x| \gg 1$, then Assumption B (2) holds with $f(r) = (\int_0^r (1 + a_1 \sin(a_2 \log t))^{-1} dt)^2$. For more examples, we refer to [10, Section 2].

We now state main results.

Theorem 1.3. *Let $d \geq 2$ and P satisfy Assumptions A and B. Then, for any $T, \varepsilon > 0$ and (p, q) satisfying (1.4), there exists $C_{T, \varepsilon} > 0$ such that*

$$\|e^{-itP}u_0\|_{L_T^p L^q} \leq C_{T, \varepsilon} \|\langle D \rangle^{\frac{1}{p}(1-\frac{2}{m})+\varepsilon} u_0\|_{L^2} + C_{T, \varepsilon} \|\langle x \rangle^{\frac{1}{p}(\frac{m}{2}-1)+\varepsilon} u_0\|_{L^2} \quad (1.7)$$

For the flat case, we can remove the additional ε -loss.

Theorem 1.4. *Let $d \geq 3$ and $H = \frac{1}{2}(D - A(x))^2 + V(x)$ satisfy Assumption A. Then, for any $T > 0$ and (p, q) satisfying (1.4),*

$$\|e^{-itH}u_0\|_{L_T^p L^q} \leq C_T \|\langle D \rangle^{\frac{1}{p}(1-\frac{2}{m})} u_0\|_{L^2} + C_T \|\langle x \rangle^{\frac{1}{p}(\frac{m}{2}-1)} u_0\|_{L^2}. \quad (1.8)$$

Corollary 1.5. *In Theorem 1.4, if we, in addition, assume (1.5) and $q < \infty$ then (1.8) holds for any dimension.*

Remark 1.6. Suppose that V satisfies (1.5). Then we can assume $P \geq 1$ without loss of generality and P hence is uniformly elliptic in the sense that

$$p(x, \xi) \approx |\xi|^2 + \langle x \rangle^m,$$

where p is the full hamiltonian associated to P (modulo lower order term), i.e.,

$$p(x, \xi) = \frac{1}{2} g^{jk}(x) (\xi_j - A_j(x)) (\xi_k - A_k(x)) + V(x).$$

By the standard parametrix construction for P , we see that, for any $1 < q < \infty$ and $s \geq 0$

$$\|P^{s/2}v\|_{L^q} + \|v\|_{L^q} \approx \|\langle D \rangle^s v\|_{L^q} + \|\langle x \rangle^{ms/2} v\|_{L^q}, \quad v \in C_0^\infty(\mathbb{R}^d),$$

(see, e.g., [38, Lemma 2.4]). The right hand side of (1.7) (resp. (1.8)) is thus dominated by $\|\langle P \rangle^{(1/2-1/m)/p+\varepsilon} u_0\|_{L^2}$ (resp. $\|\langle H \rangle^{(1/2-1/m)/p} u_0\|_{L^2}$). Therefore, our result is a generalization and improvement of Theorem 1.1.

Remark 1.7. The additional ε -loss in (1.7) is only due to the use of the following local smoothing effect:

$$\|\langle x \rangle^{-1/2-\varepsilon} E_{1/m} e^{-itP} u_0\|_{L_T^2 L^2} \leq C_{T, \varepsilon} \|u_0\|_{L^2}, \quad \varepsilon > 0,$$

where E_s is a pseudodifferential operator with the symbol $(1 + |\xi|^2 + |x|^m)^{s/2}$. It is well known that this estimate does not hold when $\varepsilon = 0$ even for $P = \frac{1}{2}\Delta + \langle x \rangle^m$ (see [25]).

Remark 1.8. It is well known that the condition $V \geq -C\langle x \rangle^2$ with some $C > 0$ is almost optimal for the essential self-adjointness of \tilde{P} . However, it was shown in [16] that \tilde{P} can be essentially self-adjoint even if V blows up in the negative direction such as $V \leq -C\langle x \rangle^m$ with $m > 2$, if strongly divergent magnetic fields are present near infinity.

Global-in-time Strichartz estimates, that is (1.3) with $T = \infty$ and $\gamma = 0$, for the free propagator $e^{it\Delta/2}$ were first proved by Strichartz [33] for a restricted pair of (p, q) with $p = q = 2(d+2)/d$, and have been generalized for (p, q) satisfying (1.4) and $p \neq 2$ by [13]. The endpoint estimate $(p, q) = (2, 2d/(d-2))$ for $d \geq 3$ was obtained by [18].

Furthermore, Strichartz estimates for Schrödinger equations have been extensively studied by many authors for both cases with potential and metric perturbations, separately.

For Schrödinger operators with potentials satisfying Assumption A with $m \leq 2$, it was shown by [12, 36] that $e^{-itH}\varphi$ satisfies (short-time) dispersive estimate: $\|e^{-itH}\varphi\|_{L^\infty} \leq C|t|^{-d/2}\|\varphi\|_{L^1}$ for sufficiently small $t \neq 0$. The estimates (1.3) with $\gamma = 0$ are immediate consequences of this estimate and the TT^* -argument due to [13] (see [18] for the endpoint). For the case with singular electric potentials, we refer to [35]. We mention that global-in-time dispersive and Strichartz estimates for scattering states have been also studied under suitable decaying conditions on potentials and assumptions for zero energy; see [17, 37, 28] for dispersive estimates and [27, 6, 9] for Strichartz estimates, and reference therein. We also mention that there is no result on sharp global-in-time dispersive estimates for (generic) magnetic Schrödinger operators, though [11] has recently proved dispersive estimates for the Aharonov-Bohm effect in \mathbb{R}^2 .

On the other hand, it is important to study the influence of underlying classical dynamics on the behavior of solutions to linear and nonlinear partial differential equations. From this geometric viewpoint, local-in-time Strichartz estimates for metric perturbations (or, more generally, on manifolds) have recently been investigated by many authors under several conditions on the geometry; see, *e.g.*, [31, 4, 24, 14, 2, 1, 5, 21] and reference therein. We mention that there are also several works on global-in-time Strichartz estimates in the case of long-range perturbations of the flat Laplacian on \mathbb{R}^d ([3, 34, 20]).

The main purpose of this paper is to handle the mixed case, especially the case for metric perturbations with unbounded electromagnetic potentials. In the previous works [22, 23], we proved the same local-in-time Strichartz estimates as in the free case under Assumptions A and B with $m < 2$ and the following long-range condition

$$|\partial_x^\alpha(g^{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x \rangle^{-\mu-|\alpha|}, \quad \mu > 0.$$

This paper is a natural continuation of this work, and the results in the series of works can be regarded as a generalization and unification of many of known local-in-time Strichartz estimates for Schrödinger equations with both of metric and unbounded potential perturbations, at least under the nontrapping condition.

1.1 Notations

Throughout the paper we use the following notations: We write $L^q = L^q(\mathbb{R}^d)$ if there is no confusion. $W^{s,q} = W^{s,q}(\mathbb{R}^d)$ is the Sobolev space with the norm $\|f\|_{W^{s,q}} = \| \langle D \rangle^s f \|_{L^q}$. For Banach spaces X and Y , $\|\cdot\|_{X \rightarrow Y}$ denotes the operator norm from X to Y . For constants $A, B \geq 0$, $A \lesssim B$ means that there exists some universal constant $C > 0$ such that $A \leq CB$. $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

We always use the letter P (resp. H) to denote variable coefficient (resp. flat) Schrödinger operators. For $h \in (0, 1]$, we consider $P^h := h^2 P$ as a semiclassical Schrödinger operator with h -dependent potentials $h^2 V$ and hA_j . We set two corresponding h -dependent symbols p^h and p_1^h defined by

$$\begin{aligned} p^h(x, \xi) &= \frac{1}{2} g^{jk}(x) (\xi_j - hA_j(x)) (\xi_k - hA_k(x)) + h^2 V(x), \\ p_1^h(x, \xi) &= -\frac{i}{2} \frac{\partial g^{jk}}{\partial x_j}(x) (\xi_k - hA_k(x)) - \frac{ih}{2} g^{jk}(x) \frac{\partial A_k}{\partial x_j}(x). \end{aligned} \tag{1.9}$$

It is easy to see that $P^h = p^h(x, hD) + hp_1^h(x, hD)$ and that Assumption A implies

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta p^h(x, \xi)| &\leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|} (|\xi|^2 + h^2 \langle x \rangle^m), \\ |\partial_x^\alpha \partial_\xi^\beta p_1^h(x, \xi)| &\leq C_{\alpha\beta} \langle x \rangle^{-1-|\alpha|} \langle \xi \rangle^{-|\beta|} (|\xi| + h \langle x \rangle^{m/2}). \end{aligned} \tag{1.10}$$

1.2 Strategy of the proof

Before starting the details of the proof, we here explain the basic idea. For simplicity we may assume $d \geq 3$. The strategy is based on microlocal techniques and basically follows the same general lines in [31, 4] and [2, Sections 5 and 6]. We however note that, since the potential V is not bounded below, the Littlewood-Paley theory in terms of the spectral multiplier $f(H)$ does not work well. For example, the Littlewood-Paley estimates of the forms

$$\|v\|_{L^q} \lesssim \|v\|_{L^2} + \left(\sum_{j=0}^{\infty} \|f(2^{2j}H)v\|_{L^q}^2 \right)^{1/2}, \quad f \in C_0^\infty(\mathbb{R}^d \setminus \{0\}), \quad (1.11)$$

seem to be false except for the trivial case $q = 2$. In particular, it is difficult to apply the localization technique due to [4] directly.

To overcome this difficulty, we introduce a partition of unity $\psi_0 + \psi_1 = 1$ on the phase space $T^*\mathbb{R}^d \cong \mathbb{R}_x^d \times \mathbb{R}_\xi^d$ with symbols $\psi_j \in S(1, dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2)$ supported in the following “high frequency” and “low frequency” regions, respectively:

$$\text{supp } \psi_0 \subset \{\langle x \rangle^m \ll |\xi|^2\}, \quad \text{supp } \psi_1 \subset \{|\xi|^2 \lesssim \langle x \rangle^m\}.$$

Let $c > 1$ and consider a c -adic partition of unity: $\theta_0, \theta \in C_0^\infty(\mathbb{R})$, $0 \leq \theta_0, \theta \leq 1$, $\text{supp } \theta \subset (1/c, c)$, $\theta_0(z) + \sum_{j=0}^{\infty} \theta(c^{-j}z) = 1$. We use this decomposition with $z = \xi, c = 2$ in the high frequency region and with $z = x, c = 2^{2/m}$ in the low frequency region, respectively. More precisely, setting $h = 2^j$ and using support properties

$$\begin{aligned} \text{supp } \theta(\xi)\psi_0(x, \xi/h) &\subset \{\langle x \rangle \lesssim h^{-2/m}, |\xi| \approx 1\}, \\ \text{supp } \theta(h^{2/m}x)\psi_1(x, \xi/h) &\subset \{\langle x \rangle \approx h^{-2/m}, |\xi| \lesssim 1\}, \end{aligned}$$

we will prove the following Littlewood-Paley type estimates:

$$\|v\|_{L^q} \lesssim \|v\|_{L^2} + \sum_{k=0,1} \left(\sum_{j=0}^{\infty} \|\Psi_k^h(x, hD)v\|_{L^q}^2 \right)^{1/2}, \quad q \in [2, \infty),$$

where $\Psi_0^h(x, \xi) = \theta(\xi)\psi_0(x, \xi/h)$ up to $O(h^\infty)$ and $\Psi_1^h(x, \xi) = \theta(h^{2/m}x)\psi_1(x, \xi/h)$. Using support properties of Ψ_k^h , we obtain the following bounds of the commutators:

$$[P, \Psi_0^h(x, hD)] = O(\langle x \rangle^{-1}h^{-1}), \quad [P, \Psi_1^h(x, hD)] = O(h^{-1+2/m}). \quad (1.12)$$

These terms can be controlled by the local smoothing effect, and the proof of Theorem 1.3 thus is reduced to that of the estimates for the localized propagators $\Psi_k^h(x, hD)e^{-itP}$.

Then, we show that, for any $\chi^h \in S(1, dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2)$ supported in $\{\langle x \rangle \lesssim h^{-2/m}, |\xi| \lesssim 1\}$ ($\supset \text{supp } \Psi_k^h$), $\chi^h(x, hD)e^{-itP}\chi^h(x, hD)^*$ satisfies the following dispersive estimates

$$\|\chi^h(x, hD)e^{-itP}\chi^h(x, hD)^*\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-d/2}, \quad |t| \ll hR, \quad h \in (0, 1], \quad (1.13)$$

where $R = \inf |\pi_x(\text{supp } \chi^h)| + 1$ for general cases and $R = h^{-2/m}$ for the flat case. These estimates are verified by a slightly refinement of the standard semiclassical parametrix construction. Namely, after rescaling $t \mapsto th$ and putting $P^h = h^2P$, we construct the semiclassical WKB parametrix for $e^{-itP^h/h}\chi^h(x, hD)^*$ of the following form:

$$e^{-itP^h/h}\chi^h(x, hD)^* = J_{S^h}(a^h) + O_{L^2 \rightarrow L^2}(h^\infty), \quad |t| \ll R,$$

where $J_{S^h}(a^h)$ is a time-dependent semiclassical Fourier integral operator

$$J_{S^h}(a^h)u_0(x) = (2\pi h)^{-d} \int e^{i(S^h(t,x,\xi) - y \cdot \xi)/h} a^h(t, x, \xi) u_0(y) dy d\xi.$$

Here the phase function $S^h(t, x, \xi)$ solves the Hamilton-Jacobi equation associated to p^h on a neighborhood of $\text{supp } f \circ p^h$ and satisfies

$$S^h(t, x, \xi) = x \cdot \xi - tp^h(x, \xi) + O(R^{-1}|t|^2), \quad |t| \ll R.$$

The amplitude a^h approximately solves the transport equation generated by the vector field $\partial_\xi p^h(x, \partial_x S^h)$ and belongs to $S(1, dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2)$ with uniform bounds in h and t . The stationary phase method then yields

$$\|J_{S^h}(a^h)\|_{L^1 \rightarrow L^\infty} \lesssim \min(h^{-d}, |th|^{-d/2}), \quad |t| \ll R,$$

which, together with a suitable error estimate, implies estimates (1.13).

Once we obtain dispersive estimates (1.13), a standard technique due to Staffilani-Tataru [31], the TT^* -argument due to Keel-Tao [18] and (1.12), we can prove the following Strichartz estimates with inhomogeneous error terms

$$\begin{aligned} & \|\Psi_k^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^{\frac{2d}{d-2}}} \\ & \lesssim \|\Psi_k^h(x, hD)u_0\|_{L^2} + \|\langle x \rangle^{-1/2} h^{-1/2} \Psi_k^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^2} \end{aligned} \quad (1.14)$$

uniformly in $h \in (0, 1]$. Combining with an almost orthogonality of $\theta(c^{-j} \cdot)$, we have

$$\|e^{-itP}u_0\|_{L_T^2 L^{\frac{2d}{d-2}}} \lesssim \|u_0\|_{L^2} + \|\langle x \rangle^{-1/2 - m\varepsilon/2} E_{1/2+\varepsilon} e^{-itP}u_0\|_{L_T^2 L^2}, \quad \varepsilon \geq 0.$$

Under Assumption B, we then use the local smoothing effect due to [25]:

$$\|\langle x \rangle^{-1/2-\nu} E_{1/m+s} e^{-itP}u_0\|_{L_T^2 L^2} \leq C_{T,\nu} \|E_s u_0\|_{L^2}, \quad \nu > 0,$$

and obtain $\|e^{-itP}u_0\|_{L_T^2 L^{\frac{2d}{d-2}}} \lesssim \|E_{1/2-1/m+\varepsilon} u_0\|_{L^2}$ if $\varepsilon > 0$. Finally, Theorem 1.3 is verified by interpolation with the trivial $L_T^\infty L^2$ -bound. In the flat case, the last term of (1.14) can be replaced by $\|h^{-1/2+1/m} \Psi_k^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^2}$ due to the fact that we obtain (1.13) with $R = h^{-2/m}$. Therefore, using the following energy estimate

$$\|E_s e^{-itP}u_0\|_{L^2} \leq C^{|t|} \|E_s u_0\|_{L^2}$$

instead of the local smoothing effect, one can remove the additional ε -loss.

The paper is organized as follows: We first record some known results on the semiclassical pseudodifferential calculus and prove the above Littlewood-Paley estimates in Section 2. Section 2 also discusses local smoothing effect and energy estimates as above. In Section 3, we construct the WKB parametrix and prove dispersive estimates (1.13). Proofs of Theorems 1.3 and 1.4 are given in Section 4. We finally prove Corollary 1.5 in Appendix A with a simpler proof than that of main theorems.

2 Preliminaries

In this section, we record some known results on the semiclassical pseudodifferential calculus and the Littlewood-Paley theory. This section also discuss local smoothing effects for the propagator e^{-itP} under Assumption B.

First of all we collect basic properties of the semiclassical pseudodifferential operator (h - Ψ DO for short). We omit proofs and refer to [26, 19] for the details. Set a metric on the phase space $T^*\mathbb{R}^d \cong \mathbb{R}^{2d}$ defined by $g = dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2$. For a g -continuous weight function $m(x, \xi)$, we use Hörmander's symbol class $S(m, g)$, which is the space of smooth functions on \mathbb{R}^{2d} satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|}.$$

To a symbol $a \in C^\infty(\mathbb{R}^{2d})$ and $h \in (0, 1]$, we associate the h - Ψ DO $a(x, hD)$ defined by

$$a(x, hD)f(x) = (2\pi h)^{-d} \int e^{i(x-y)\cdot\xi/h} a(x, \xi) f(y) dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class. For a h - Ψ DO A , we denote its symbol by $\text{Sym}(A)$, *i.e.*, $A = a(x, hD)$ if $a = \text{Sym}(A)$. It is known as the Calderón-Vaillancourt theorem that for any symbol $a \in C^\infty(\mathbb{R}^{2d})$ satisfying $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta}$, $a(x, hD)$ is extended to a bounded operator on $L^2(\mathbb{R}^d)$ with a uniform bound in $h \in (0, 1]$. Moreover, if $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{-\gamma}$ with some $\gamma > d$, then $a(x, hD)$ is extended to a bounded operator from L^q to L^r with bounds

$$\|a(x, hD)\|_{L^q \rightarrow L^r} \leq C_{qr} h^{-d(1/q-1/r)}, \quad 1 \leq q \leq r \leq \infty, \quad (2.1)$$

where $C_{qr} > 0$ is independent of $h \in (0, 1]$. These bounds follow from the Schur lemma and the Riez-Thorin interpolation theorem (see, *e.g.*, [2, Proposition 2.4]). For two symbols $a \in S(m_1, g)$ and $b \in S(m_2, g)$, the composition $a(x, hD)b(x, hD)$ is also a h - Ψ DO with the symbol $a \sharp b(x, \xi) = e^{ihD_\eta D_z} a(x, \eta) b(z, \xi)|_{z=x, \eta=\xi} \in S(m_1 m_2, g)$, which has the expansion

$$a \sharp b - \sum_{|\alpha| < N} \frac{h^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_\xi^\alpha a \cdot \partial_x^\alpha b \in S(h^N \langle x \rangle^{-N} \langle \xi \rangle^{-N} m_1 m_2, g). \quad (2.2)$$

In particular, we have $\text{Sym}([a(x, hD), b(x, hD)]) - \frac{h}{i} \{a, b\} \in S(h^2 \langle x \rangle^{-2} \langle \xi \rangle^{-2}, g)$, where $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b$ is the Poisson bracket. The symbol of the adjoint $a(x, hD)^*$ is given by $a^*(x, \xi) = e^{ihD_\eta D_z} a(z, \eta)|_{z=x, \eta=\xi} \in S(m_1, g)$ which has the expansion

$$a^* - \sum_{|\alpha| < N} \frac{h^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_\xi^\alpha \partial_x^\alpha a \in S(h^N \langle x \rangle^{-N} \langle \xi \rangle^{-N} m_1, g). \quad (2.3)$$

We also often use the following which is a direct consequence of (2.2):

Lemma 2.1. *Let $a \in S(m_1, g)$ and $b \in S(m_2, g)$. If $b \equiv 1$ on $\text{supp } a$, then for any $N \geq 0$,*

$$a(x, hD) = a(x, hD)b(x, hD) + h^N r_N(x, hD) = b(x, hD)a(x, hD) + h^N \tilde{r}_N(x, hD)$$

with some $r_N, \tilde{r}_N \in S(\langle x \rangle^{-N} \langle \xi \rangle^{-N} m_1 m_2, g)$.

2.1 Littlewood-Paley estimates

We here prove the following Littlewood-Paley type estimates, which will be used to reduce the proof of the estimates (1.7) to that of semiclassical Strichartz estimates. Here, and in what follows, the summation over h , \sum_h , means that, in the sum, h takes all negative powers of 2 as values, *i.e.*, $\sum_h := \sum_{h=2^{-j}, j \geq 0}$.

Proposition 2.2 (Littlewood-Paley estimates). *For $h \in (0, 1]$, there exist two symbols Ψ_0^h and Ψ_1^h such that the following statements are satisfied :*

(1) (Symbol estimates) $\{\Psi_k^h\}_{h \in (0, 1]}$ are bounded in $S(1, h^{4/m} dx^2 + d\xi^2 / \langle \xi \rangle^2)$, i.e.,

$$|\partial_x^\alpha \partial_\xi^\beta \Psi_k^h(x, \xi)| \leq C_{\alpha\beta} h^{(2/m)|\alpha|} \langle \xi \rangle^{-|\beta|}, \quad k = 0, 1,$$

uniformly in $h \in (0, 1]$.

(2) (Support property) Ψ_k^h satisfy the following support properties (uniformly in h):

$$\text{supp } \Psi_0^h \subset \{(x, \xi); h^2 \langle x \rangle^m \lesssim 1, |\xi|^2 \approx 1\}, \quad (2.4)$$

$$\text{supp } \Psi_1^h \subset \{(x, \xi); h^2 \langle x \rangle^m \approx 1, |\xi|^2 \lesssim 1\}. \quad (2.5)$$

(3) (Littlewood-Paley estimates) For any $q \in [2, \infty)$,

$$\|v\|_{L^q} \lesssim \|v\|_{L^2} + \sum_{k=0,1} \left(\sum_h \|\Psi_k^h(x, hD)v\|_{L^q}^2 \right)^{1/2}, \quad (2.6)$$

where the implicit constant is independent of v .

Remark 2.3. As we mentioned in the strategy of the proof, it seems to be difficult to obtain the Littlewood-Paley estimates (1.11) if $q \neq 2$. We, however, note that if V satisfies (1.5), then one can prove (1.11) for $q \in [2, \infty)$ (see Appendix).

In order to prove Proposition 2.2, we first construct the symbols Ψ_k^h . Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\text{supp } \varphi \subset [-1, 1]$, $\varphi \equiv 1$ on $[-1/2, 1/2]$ and $0 \leq \varphi \leq 1$. Define smooth cut-off functions into high and low frequency regions by

$$\psi_0(x, \xi) = \varphi\left(\frac{\langle x \rangle^{m/2}}{\varepsilon |\xi|}\right), \quad \psi_1 = 1 - \psi_0,$$

respectively, where $\varepsilon > 0$ is a sufficiently small constant such that $p(x, \xi) \approx |\xi|^2$ if $\langle x \rangle^m \leq \varepsilon |\xi|^2$. It is easy to see that $\text{supp } \psi_0 \subset \{(x, \xi); \langle x \rangle^m \leq \varepsilon^2 |\xi|^2\}$, $\text{supp } \psi_1 \subset \{(x, \xi); \langle x \rangle^m \geq \varepsilon^2 |\xi|^2/2\}$ and that $\psi_0, \psi_1 \in S(1, g)$ for each $\varepsilon > 0$.

Lemma 2.4. *For any $\theta \in C_0^\infty(\mathbb{R}^d)$ supported away from the origin and any $N > d$, there exists a bounded family $\{\Psi_0^h\}_{h \in (0, 1]} \subset S(1, h^{4/m} dx^2 + d\xi^2 / \langle \xi \rangle^2)$ satisfying (2.4) such that*

$$\|\theta(hD)\psi_0(x, D) - \Psi_0^h(x, hD)\|_{L^2 \rightarrow L^q} \leq C_{qN} h^{N-d(1/2-1/q)}, \quad q \in [2, \infty),$$

where $C_{qN} > 0$ may be taken uniformly in $h \in (0, 1]$. Moreover, if we set

$$\Psi_1^h(x, \xi) := \theta(h^{m/2}x)\psi_1(x, \xi/h),$$

then $\{\Psi_1^h\}_{h \in (0, 1]}$ is bounded in $S(1, h^{4/m} dx^2 + d\xi^2 / \langle \xi \rangle^2)$ and satisfies the support property (2.5).

Proof. Choose $\tilde{\theta} \in C_0^\infty(\mathbb{R}^d)$ so that $\tilde{\theta}$ is supported away from the origin and that $\tilde{\theta} \equiv 1$ on $\text{supp } \theta$. Then we learn by the expansion formula (2.2) (with $h = 1$) that

$$\theta(hD)\psi_0(x, D) = \theta(hD)\tilde{\theta}(hD)\psi_0(x, D) = \theta(hD)\tilde{\psi}_0^h(x, D) + \theta(hD)\tilde{r}_N^h(x, D),$$

where $\tilde{\psi}_0^h \in S(1, g)$ is given by

$$\tilde{\psi}_0^h(x, \xi) = \sum_{|\alpha| < N} \frac{i^{-|\alpha|}}{\alpha!} \partial_\eta^\alpha \partial_z^\alpha \tilde{\theta}(h\eta)\psi_0(z, \xi) \Big|_{\eta=\xi, z=x} \quad (2.7)$$

and $\tilde{r}_N^h \in S(\langle x \rangle^{-N} \langle \xi \rangle^{-N}, g)$ with uniform bounds in h . Since $|\xi| \approx h^{-1}$ on $\text{supp } \theta(h\xi)$, (2.1) with $h = 1$ and Bernstein's inequality imply

$$\|\theta(hD)\tilde{r}_N^h(x, D)\|_{L^2 \rightarrow L^q} \leq \|\theta(hD)\langle D \rangle^{-N}\|_{L^2 \rightarrow L^q} \|\langle D \rangle^N \tilde{r}_N^h(x, D)\|_{L^2 \rightarrow L^2} \lesssim h^{N-d(1/2-1/q)}.$$

We next consider the first term. Since

$$\text{supp } \tilde{\psi}_0^h(\cdot, \cdot/h) \subset \{(x, \xi) \in \text{supp } \psi_0(\cdot, \cdot/h); \xi \in \text{supp } \theta\} \subset \{(x, \xi); h^2 \langle x \rangle^m \lesssim 1, |\xi| \approx 1\},$$

we have

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{\psi}_0^h(x, \xi/h)| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} h^{-|\beta|} \langle \xi/h \rangle^{-|\beta|} \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} (h + |\xi|)^{-|\beta|} \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle |\xi| \rangle^{-|\beta|}.$$

Therefore $\{\tilde{\psi}_0^h(\cdot, \cdot/h)\}_{h \in (0,1]}$ is bounded in $S(1, g)$, while $\psi_0(x, \xi/h)$ may have singularities at $\xi = 0$ as $h \rightarrow 0$. In particular, $\tilde{\psi}_0^h(x, D)$ can be regarded as a h -ΨDO with the symbol $\tilde{\psi}_0^h(\cdot, \cdot/h)$. (2.2) again implies that there exist bounded families $\{\Psi_0^h\}_{h \in (0,1]} \subset S(1, g)$ and $\{r_N^h\}_{h \in (0,1]} \subset S(\langle x \rangle^{-N} \langle \xi \rangle^{-N}, g)$ such that

$$\theta(hD)\tilde{\psi}_0^h(x, D) = \Psi_0^h(x, hD) + h^N r_N^h(x, hD), \quad \|r_N^h(x, hD)\|_{L^2 \rightarrow L^q} \leq C_{qN} h^{-d(1/2-1/q)}.$$

Moreover, Ψ_0^h is given explicitly by

$$\Psi_0^h(x, \xi) = \sum_{|\alpha| < N} \frac{h^{|\alpha|} i^{-|\alpha|}}{\alpha!} \partial_\eta^\alpha \partial_z^\alpha \theta(\eta) \tilde{\psi}_0^h(z, \xi/h) \Big|_{\eta=\xi, z=x}. \quad (2.8)$$

By virtue of (2.7) and (2.8), we see that

$$\text{supp } \Psi_0^h \subset \text{supp } \theta \cap \text{supp } \psi_0(\cdot, \cdot/h) \subset \{h^2 \langle x \rangle^m \lesssim |\xi|^2, |\xi| \approx 1\}$$

and that, for $j = 1, 2, \dots, d$,

$$\text{supp } \partial_{x_j} \Psi_0^h \subset \text{supp } \theta \cap \text{supp } \psi'_0(\cdot, \cdot/h) \subset \{h^2 \langle x \rangle^m \approx |\xi|^2, |\xi| \approx 1\}$$

Therefore, Ψ_0^h satisfies (2.4) and

$$|\partial_x^\alpha \partial_\xi^\beta \Psi_k^h(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle x \rangle^{-|\beta|} \leq C_{\alpha\beta} h^{2|\alpha|/m} \langle \xi \rangle^{-|\beta|}, \quad h \in (0, 1].$$

Finally, since $\partial_x^\alpha \partial_\xi^\beta \psi_1$ are supported in $\text{supp } \psi'_0$ for any $|\alpha + \beta| \geq 1$, we learn $|\xi| \approx h^2 \langle x \rangle^m \approx 1$ on $\text{supp } \theta(h^{2/m} x) \cap \text{supp } \partial_x^\alpha \partial_\xi^\beta \psi_1(x, \xi/h)$ as long as $|\alpha + \beta| \geq 1$. Hence $\{\Psi_1^h\}_{h \in (0,1]}$ is also bounded in $S(1, h^{4/m} dx^2 + d\xi^2 / \langle \xi \rangle^2)$ and satisfies the support property (2.5). \square

We next recall the square function estimates for the standard Littlewood-Paley projections.

Lemma 2.5 (Square function estimates). *Let $c > 1$ and consider a c -adic partition of unity:*

$$\theta_0, \theta \in C_0^\infty(\mathbb{R}^d), \text{supp } \theta \subset \{1/c < |x| < c\}, \quad 0 \leq \theta_0, \theta \leq 1, \quad \theta_0(x) + \sum_{l \geq 0} \theta(c^{-l} x) = 1.$$

Then, for any $1 < q < \infty$,

$$\|v\|_{L^q} \approx \left\| \left(|\theta_0(D)v|^2 + \sum_{l \geq 0} |\theta(c^{-l} D)v|^2 \right)^{1/2} \right\|_{L^q} \approx \left\| \left(|\theta_0(x)v|^2 + \sum_{l \geq 0} |\theta(c^{-l} x)v|^2 \right)^{1/2} \right\|_{L^q}.$$

Moreover, if $2 \leq q < \infty$ then

$$\|v\|_{L^q} \lesssim \|v\|_{L^2} + \left(\sum_l \|\theta(c^{-l} D)v\|_{L^q}^2 \right)^{1/2}, \quad (2.9)$$

$$\|v\|_{L^q} \lesssim \|\theta_0(x)v\|_{L^q} + \left(\sum_l \|\theta(c^{-l} x)v\|_{L^q}^2 \right)^{1/2}. \quad (2.10)$$

Proof. We let $q \in (1, \infty)$ and set $S_1 v = (|\theta_0(x)v|^2 + \sum_{l \geq 0} |\theta(c^{-l}x)v|^2)^{1/2}$. Since $\theta^2 \leq \theta$,

$$\left(\sum_{l \geq 0} |\theta(c^{-l}x)|^2 \right)^{1/2} \leq \left(\sum_{l \geq 0} \theta(c^{-l}x) \right)^{1/2} = (1 - \theta_0)^{1/2} \leq 1,$$

from which we have $\|S_1 v\|_{L^q} \leq 2\|v\|_{L^q}$. Since $\theta(c^{-l}x)\theta(c^{-k}x) = 0$ for $|l - k| > 2$, we learn by Hölder's inequality that

$$\left| \int v_1 \overline{v_2} dx \right| \lesssim \|S_1 v_1\|_{L^q} \|S_1 v_2\|_{L^{q'}}, \quad 1/q + 1/q' = 1,$$

which, together with the upper bound, implies $\|v\|_{L^q} \lesssim \|S_1 v\|_{L^q}$. Therefore, $\|v\|_{L^q} \approx \|S_1 v\|_{L^q}$. On the other hand, the first estimate can be proved by a similar argument, combined with the L^q -boundedness of the Fourier multiplier $T_t v = \sum_{l \geq 0} r_l(t) \theta(c^{-l}D)v$ (see, e.g., [30] for the details), where $r_l(t)$ are Rademacher functions. (2.9) and (2.10) then follows from Minkowski's inequality and Bernstein's inequality since $q \geq 2$. \square

We now turn into the proof of Proposition 2.2:

Proof of Proposition 2.2. Set $h = 2^{-l}$. We plug $\psi_0(x, D)v$ into (2.9) with $c = 2$. By virtue of Lemma 2.4, the contribution of the error term $\theta(hD)\tilde{r}_N^h(x, D) + h^N r_N^h(x, hD)$ is dominated by $\|v\|_{L^2}$ provided that $N > d(1/2 - 1/q)$. We hence have

$$\|\psi_0(x, D)v\|_{L^q} \lesssim \|v\|_{L^2} + \left(\sum_h \|\Psi_0^h(x, hD)v\|_{L^q}^2 \right)^{1/2}.$$

The estimate for $\psi_1(x, D)v$ is verified similarly by using Lemma 2.4 and (2.10) with $c = 2^{2/m}$. \square

2.2 Local smoothing effects

We here recall the local smoothing effects proved by Robbiano-Zuily [25]. For $s \in \mathbb{R}$ we set

$$e_s(x, \xi) := (k_A(x, \xi) + \langle x \rangle^m + L(s))^{s/2},$$

where $k_A(x, \xi) = \frac{1}{2}g^{jk}(x)(\xi_j - A_j(x))(\xi_k - A_k(x))$ and $L(s)$ is a constant depending on s . Then, $e_s \in S(e_s, dx^2/\langle x \rangle^2 + d\xi^2/e_1^2)$, that is

$$|\partial_x^\alpha \partial_\xi^\beta e_s(x, \xi)| \leq C_{\alpha\beta} e_{s-|\beta|}(x, \xi) \langle x \rangle^{-|\alpha|}. \quad (2.11)$$

Let $E_s = e_s(x, D)$ and $\mathcal{B}^s := \{f; \langle x \rangle^s f \in L^2, \langle D \rangle^2 f \in L^2\}$. Then, for any $s \in \mathbb{R}$, there exists $L(s) > 0$ such that E_s is a homeomorphism from \mathcal{B}^{r+s} to \mathcal{B}^r for all $r \in \mathbb{R}$, and E_s^{-1} is also a Ψ DO with the symbol \tilde{e}_{-s} in $S(e_{-s}, dx^2/\langle x \rangle^2 + d\xi^2/e_1^2)$ (see, [10, Lemma 4.1]).

We first prepare the following two lemmas:

Lemma 2.6. *For any $s \in \mathbb{R}$, $E_s P E_s^{-1} = P + B_s$ with $\|B_s - B_s^*\|_{L^2 \rightarrow L^2} \lesssim 1$.*

Proof. We write $P = p_{\text{full}}(x, D) = k_A(x, D) + p_1(x, D) + V(x)$, where

$$p_1(x, \xi) = -\frac{i}{2} \frac{\partial g^{jk}}{\partial x_j}(x)(\xi_k - A_k(x)) - \frac{i}{2} g^{jk}(x) \frac{\partial A_k}{\partial x_j}(x).$$

A direct computation yields

$$\{e_s, k_A + V\} = -\frac{s}{2} e_{s-2} \{\langle x \rangle^m, k_A\} + \frac{s}{2} e_{s-2} \{k_A, V\} = e_{s-2} (a_0 + a_1)$$

where $a_0 \in S(\langle x \rangle^{m-1} \langle \xi \rangle, g)$, and $a_1(x)$ is independent of ξ and satisfies $\partial_x^\alpha a_1 = O(\langle x \rangle^{3m/2-1-|\alpha|})$. We similarly obtain $\{e_s, p_1\} \in S(e_s \langle x \rangle^{-2}, g)$ by (2.11). Next, since (1.1) and (2.11) yield

$$\partial_\xi^\alpha e_s \in S(e_{s-2}, g), \quad \partial_x^\alpha e_s \in S(e_s, g), \quad \partial_\xi^\alpha p_{\text{full}} \in S(1, g), \quad \partial_x^\alpha p_{\text{full}} \in S(e_2, g), \quad |\alpha| \geq 2,$$

the expansion formula (2.2) shows that the symbol of $[E_s, P] - \{e_s, p_{\text{full}}\}(x, D)$ belongs to $S(e_s, g)$. We also learn by (2.11) that

$$\begin{aligned} \partial_\xi^\alpha (e_{s-2} a_0) \partial_x^\alpha \tilde{e}_{-s} &= O(e_{-3} \langle x \rangle^{m-1} \langle \xi \rangle + e_{-2} \langle x \rangle^{m-1}) = O(\langle x \rangle^{-1}), \\ \partial_\xi^\alpha (e_{s-2} a_1) \partial_x^\alpha \tilde{e}_{-s} &= O(e_{-3} \langle x \rangle^{3m/2-1}) = O(\langle x \rangle^{-1}), \end{aligned}$$

for all $|\alpha| \geq 1$. Therefore, the symbol of $B_s = [E_s, P]E_s^{-1}$ is of the form $e_{s-2} \tilde{e}_{-s}(a_0 + a_1) + a_2$ with $a_2 \in S(1, g)$. By a similar argument, we further have

$$\partial_\xi^\alpha \partial_x^\alpha (e_{s-2} \tilde{e}_{-s} a_0) = O(\langle x \rangle^{-1}), \quad \partial_\xi^\alpha \partial_x^\alpha (e_{s-2} \tilde{e}_{-s} a_1) = O(\langle x \rangle^{-1}), \quad |\alpha| \geq 1,$$

which, together with the expansion formula (2.3), imply that the symbol of $B_s - B_s^*$ belongs to $S(1, g)$. Then, the assertion is a consequence of the Calderón-Vaillancourt theorem. \square

Lemma 2.7. *For any $s \in \mathbb{R}$ there exists $C_s > 0$ such that*

$$\|E_s e^{-itP} u_0\|_{L^2} \leq C_s e^{C_s |t|} \|E_s u_0\|_{L^2}, \quad t \in \mathbb{R}.$$

Proof. We may assume $t \geq 0$ without loss of generality. Set $v(t) = E_s e^{-itP} u_0$ and compute

$$\frac{d}{dt} \|v(t)\|_{L^2}^2 = \langle -i(P + B_s)v(t), v(t) \rangle + \langle v(t), -i(P + B_s)v(t) \rangle = -i\langle (B_s - B_s^*)v(t), v(t) \rangle.$$

By virtue of the previous lemma, we have $\frac{d}{dt} \|v(t)\|_{L^2} \lesssim \|v(t)\|_{L^2}$. The assertion then follows from Gronwall's inequality. \square

We now state the local smoothing effects for the propagator e^{-itP} .

Proposition 2.8 (The local smoothing effects [25]). *Suppose Assumptions A and B. Then, for any $T > 0$, $\nu > 0$ and $s \in \mathbb{R}$, there exists $C_{T,\nu,s} > 0$ such that*

$$\|\langle x \rangle^{-1/2-\nu} E_{s+1/m} e^{-itP} u_0\|_{L_T^2 L^2} \leq C_{T,\nu,s} \|E_s u_0\|_{L^2}. \quad (2.12)$$

Proof. Robbiano-Zuliy [25] proved the case when $s = 0$ only. However, by virtue of Lemmas 2.6 and 2.7, general cases can be verified by an essentially same argument. We refer to [10, Section 8] in which one can find the details of the proof for the case with $m = 2$ and $s \in \mathbb{R}$. \square

Remark 2.9. Assumption B is only needed for Proposition 2.8.

3 Parametrix construction

Write $\Gamma^h(L) := \{(x, \xi); |\xi|^2 + h^2 \langle x \rangle^m < L\}$, where $L \geq 1$ is a large such that $\text{supp } \Psi_k^h \subset \Gamma^h(L)$, $k = 0, 1$. This section is devoted to construct the parametrices of propagators, localized in this energy shell, in terms of the semiclassical Fourier integral operator (h -FIO for short).

3.1 Classical mechanics

We here study behaviors of the Hamilton flow generated by p^h . Consider the following Hamilton system:

$$\begin{cases} \dot{X}_j = \frac{\partial p^h}{\partial \xi_j}(X, \Xi) = g^{jk}(X)(\Xi_k - hA_k(X)), \\ \dot{\Xi}_j = -\frac{\partial p^h}{\partial x_j}(X, \Xi) = -\frac{1}{2} \frac{\partial g^{kl}}{\partial x_j}(X)(\Xi_k - hA_k(X))(\Xi_l - hA_l(X)) \\ \quad + hg^{kl}(X) \frac{\partial A_k}{\partial x_j}(X)(\Xi_l - hA_l(X)) - h^2 \frac{\partial V}{\partial x_j}(X), \end{cases} \quad (3.1)$$

with the initial condition $(X(0, x, \xi), \Xi(0, x, \xi)) = (x, \xi) \in \Gamma^h(L)$, where $\dot{f} = \partial_t f$. For simplicity, we here suppress the h -dependence of the flow.

We first show that the flow is well-defined for $|t| \leq \delta h^{-2/m}$ and $(x, \xi) \in \Gamma^h(L)$ with sufficiently small $\delta > 0$, which is not obvious since the potential V can blow up in the negative direction like $V(x) \leq -C\langle x \rangle^m$. More precisely, we have the following rough a priori bound:

Lemma 3.1. *For sufficiently small $\delta_0 > 0$ there exists $C = C(m, L, \delta_0) > 0$ such that*

$$|\Xi(t, x, \xi)|^2 + h^2 \langle X(t, x, \xi) \rangle^m \leq C$$

uniformly in $h \in (0, 1]$ and $(t, x, \xi) \in [-\delta_0 h^{-2/m}, \delta_0 h^{-2/m}] \times \Gamma^h(L)$.

Proof. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be such that $\rho(x) = 1$ for $|x| \leq 1$ and $\rho(x) = 0$ for $|x| \geq 2$ and set

$$\begin{aligned} \tilde{V}(x) &= \rho\left(\frac{h^{2/m}x}{2L}\right) V(x), \quad \tilde{A}_j(x) = \rho\left(\frac{h^{2/m}x}{2L}\right) A_j(x), \\ \tilde{p}^h(x, \xi) &= \frac{1}{2} g^{jk}(\xi_j - h\tilde{A}_j(x))(\xi_k - h\tilde{A}_k(x)) + h^2 \tilde{V}(x). \end{aligned} \quad (3.2)$$

Note that $h^2 |\tilde{V}(x)| + h^2 |\tilde{A}_j(x)|^2 \lesssim L$ on \mathbb{R}^d by Assumption A. Consider the corresponding Hamilton flow $(\tilde{X}(t, x, \xi), \tilde{\Xi}(t, x, \xi))$, that is the solution to

$$\dot{\tilde{X}} = \partial_\xi \tilde{p}^h(\tilde{X}, \tilde{\Xi}), \quad \dot{\tilde{\Xi}} = -\partial_x \tilde{p}^h(\tilde{X}, \tilde{\Xi}); \quad (\tilde{X}, \tilde{\Xi})|_{t=0} = (x, \xi).$$

Since the flow conserves the energy, i.e., $\tilde{p}^h(\tilde{X}(t), \tilde{\Xi}(t)) = \tilde{p}^h(x, \xi)$, we learn by the ellipticity of g^{jk} that if $(x, \xi) \in \Gamma^h(L)$, then

$$|\tilde{\Xi}|^2 \leq |\tilde{\Xi} - h\tilde{A}(\tilde{X})|^2 + L \lesssim \tilde{p}^h(x, \xi) + h^2 |\tilde{V}(\tilde{X})| + L \lesssim L,$$

and hence $|\tilde{X}| \lesssim |\tilde{\Xi} - h\tilde{A}(\tilde{X})| \leq C_1 L^{1/2}$ with some universal constant $C_1 > 0$. Therefore, for any fixed $m, L > 0$, taking $0 < \delta_0 < C_1^{-1} L^{-1/2} (2L - L^{1/m})$ we have

$$h^{2/m} |\tilde{X}| \leq h^{2/m} (|x| + C_1 L^{1/2} |t|) \leq L^{1/m} + C_1 L^{1/2} \delta_0 < 2L$$

on $[-\delta_0 h^{-2/m}, \delta_0 h^{-2/m}] \times \Gamma^h(L)$. By virtue of this bound and the fact that

$$p^h(x, \xi) \equiv \tilde{p}^h(x, \xi) \quad \text{if} \quad h^{2/m} |x| \leq 2L,$$

the uniqueness theorem of first order ODE shows $(X, \Xi) \equiv (\tilde{X}, \tilde{\Xi})$ on $[-\delta_0 h^{-2/m}, \delta_0 h^{-2/m}] \times \Gamma^h(L)$. In particular, (X, Ξ) is well defined on $[-\delta_0 h^{-2/m}, \delta_0 h^{-2/m}] \times \Gamma^h(L)$ and the above computations yield the desired bound. \square

We next study more precise behavior of the flow. Set

$$\Omega^h(R, L) := \{|x| > R\} \cap \Gamma^h(L), \quad \Omega_0^h(r, L) := \{|x| < r\} \cap \Gamma^h(L).$$

Note that $\Omega^h(R, L) \subseteq \Omega^h(R', L')$ and $\Omega_0^h(r, L) \subseteq \Omega_0^h(r', L')$ if $R > R'$, $r < r'$ and $L < L'$.

Lemma 3.2 (General case). *For sufficiently small $0 < \delta < \delta_0$, the followings are satisfied:*

(1) *For any $h \in (0, 1]$, $1 \leq R \leq h^{-2/m}$, $(t, x, \xi) \in [-\delta R, \delta R] \times \Omega^h(R, L)$,*

$$|X(t) - x| + \langle x \rangle |\Xi(t) - \xi| \leq C|t|, \quad (3.3)$$

$$|\partial_x^\alpha \partial_\xi^\beta (X(t) - x)| + \langle x \rangle |\partial_x^\alpha \partial_\xi^\beta (\Xi(t) - \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1} |t|, \quad |\alpha + \beta| \geq 1, \quad (3.4)$$

where constants $C, C_{\alpha\beta} > 0$ may be taken uniformly in h, R and t . Moreover, for fixed $h \in (0, 1]$, $1 \leq R \leq h^{-2/m}$ and $|t| \leq \delta R$, the map $\Lambda(t) : (x, \xi) \mapsto (X(t, x, \xi), \xi)$ is diffeomorphic from $\Omega^h(R/2, 2L)$ onto its range and satisfies

$$\Omega^h(R, L) \subset \Lambda(t, \Omega^h(R/2, 2L)) \subset \Omega^h(R/3, 3L), \quad h \in (0, 1], \quad |t| \leq \delta R. \quad (3.5)$$

(2) *If $(Y(t, x, \xi), \xi)$ denotes the inverse map of $\Lambda(t)$, then bounds (3.3) and (3.4) still hold with $X(t)$ replaced by $Y(t)$ for $(t, x, \xi) \in [-\delta R, \delta R] \times \Omega^h(R, L)$.*

(3) *The same conclusions also hold with $R = 1$ and with $\Omega^h(R, L)$ replaced by $\Omega_0^h(r, L)$, i.e., $X(t)$ and $Y(t)$ satisfy (3.3) and (3.4) uniformly in $h \in (0, 1]$ and $(t, x, \xi) \in [-\delta, \delta] \times \Omega_0^h(r, L)$.*

Proof. We prove the assertions (1) and (2) only since the proof of (3) being similar. Suppose that $|t| \leq \delta R$ and $(x, \xi) \in \Omega^h(R, L)$. By the Hamilton system and Lemma 3.1,

$$|\dot{X}| \lesssim 1, \quad |\dot{\Xi}| \lesssim \langle X \rangle^{-1} + h \langle X \rangle^{m/2-1} + h^2 \langle X \rangle^{m-1} \lesssim \langle X \rangle^{-1}, \quad (3.6)$$

By the first estimate, we see that if $\delta > 0$ is small enough then

$$|X(t, x, \xi)| \approx |x| \quad \text{for } (t, x, \xi) \in (-\delta R, \delta R) \times \Omega^h(R, L),$$

which, together with (3.6), implies (3.3).

We next let $|\alpha + \beta| = 1$ and differentiate the Hamilton system

$$\begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta \dot{X} \\ \langle x \rangle \partial_x^\alpha \partial_\xi^\beta \dot{\Xi} \end{pmatrix} = \begin{pmatrix} \partial_x \partial_\xi p^h(X, \Xi) & \langle x \rangle^{-1} \partial_\xi^2 p^h(X, \Xi) \\ -\langle x \rangle \partial_x^2 p^h(X, \Xi) & -\partial_\xi \partial_x p^h(X, \Xi) \end{pmatrix} \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X \\ \langle x \rangle \partial_x^\alpha \partial_\xi^\beta \Xi \end{pmatrix} = O(\langle x \rangle^{-1}) \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X \\ \langle x \rangle \partial_x^\alpha \partial_\xi^\beta \Xi \end{pmatrix}$$

where we have used the fact that

$$|(\partial_x^2 p^h)(X, \Xi)| \lesssim \langle x \rangle^{-2}, \quad |(\partial_\xi^2 p^h)(X, \Xi)| \lesssim 1, \quad |(\partial_x \partial_\xi p^h)(X, \Xi)| \lesssim \langle x \rangle^{-1}$$

on $(-\delta R, \delta R) \times \Omega^h(R, L)$. The estimate (3.4) with $|\alpha + \beta| = 1$ then follows from Gronwall's inequality. Proofs for higher derivatives follow from an induction on $|\alpha + \beta|$. The inclusion relation (3.5) and the existence of the inverse of $\Lambda^h(t)$ are verified by a standard argument based on the Hadamard global inverse mapping theorem (see, e.g., [22, Lemmas A.2 and A.4]). The estimates for $Y(t)$ are verified by differentiating the equality $x = X(t, Y(t, x, \xi))$ and using the estimates for $X(t)$. \square

When the flat case, we have the following stronger bounds than in the previous lemma:

Lemma 3.3 (Flat case). *Assume that $g^{jk} \equiv \delta_{jk}$. Then, for sufficiently small $0 < \delta < \delta_0$, the followings hold uniformly with respect to $h \in (0, 1]$:*

(1) *For any $(t, x, \xi) \in [-\delta h^{-2/m}, \delta h^{-2/m}] \times \Gamma^h(L)$, we have*

$$|X(t) - x| + h^{-2/m}|\Xi(t) - \xi| \leq C|t| \quad (3.7)$$

$$|\partial_x^\alpha \partial_\xi^\beta (X(t) - x)| \leq C_{\alpha\beta} h^{2/m} |t|, \quad |\alpha + \beta| \geq 1, \quad (3.8)$$

$$|\partial_x \Xi(t)| \leq C_\alpha h^{2/m} \langle x \rangle^{-1} |t|, \quad |\partial_\xi (\Xi(t) - \xi)| \leq C_\alpha h^{4/m} |t|, \quad (3.9)$$

$$|\partial_x^\alpha \partial_\xi^\beta (\Xi(t) - \xi)| \leq C_{\alpha\beta} h^{2/m} \langle x \rangle^{-1} |t|, \quad |\alpha + \beta| \geq 2. \quad (3.10)$$

(2) *The inverse map of $\Lambda(t)$, denoted by $(Y(t, x, \xi), \xi)$, is well-defined for $(-\delta h^{-2/m}, \delta h^{-2/m}) \times \Gamma^h(L)$ and the bounds (3.7) and (3.8) still hold with $X(t)$ replaced by $Y(t)$.*

Proof. Suppose that $(t, x, \xi) \in [-\delta h^{-2/m}, \delta h^{2/m}] \times \Gamma^h(L)$. Set $\eta(t) = \Xi(t) + hA(X(t))$. $X(t)$, $\Xi(t)$ and $\eta(t)$ then solve

$$\begin{aligned} \dot{X} &= \Xi - hA(X) = \eta - 2hA(X), \quad \dot{\Xi} = -h\partial_x A(X)(\Xi - hA(X)) - h^2 \partial_x V(X), \\ \dot{\eta} &= \dot{\Xi} + h\partial_x A(X)\dot{X} = -h^2 \partial_x V(X). \end{aligned} \quad (3.11)$$

The a priori bound in Lemma 3.1 shows

$$|\dot{X}| \lesssim 1 + h\langle X \rangle^{m/2} \lesssim 1, \quad |\dot{\Xi}| \lesssim h\langle X \rangle^{m/2-1} + h^2 \langle X \rangle^{m-1} \lesssim h^{2/m}, \quad (3.12)$$

which imply (3.7). Next, we let $|\alpha + \beta| = 1$ and learn by (3.11) and (3.7) that

$$\begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta \dot{X} \\ h^{-2/m} \partial_x^\alpha \partial_\xi^\beta \dot{\eta} \end{pmatrix} = \begin{pmatrix} -h\partial_x A(X) & h^{2/m} \\ -h^{2-2/m} \partial_x^2 V(X) & 0 \end{pmatrix} \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X \\ h^{-2/m} \partial_x^\alpha \partial_\xi^\beta \eta \end{pmatrix} = O(h^{2/m}) \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X \\ h^{-2/m} \partial_x^\alpha \partial_\xi^\beta \eta \end{pmatrix}.$$

Therefore, since $|\partial_x^\alpha \partial_\xi^\beta x| + h^{1-2/m} |\partial_x^\alpha \partial_\xi^\beta \partial_x A(x)| = O(1)$ on $\Gamma^h(L)$, we have

$$\begin{aligned} & |\partial_x^\alpha \partial_\xi^\beta (X(t) - x)| + h^{-2/m} |\partial_x^\alpha \partial_\xi^\beta (\eta(t) - h\partial_x A(x))| \\ & \lesssim h^{2/m} \int_0^t (|\partial_x^\alpha \partial_\xi^\beta (X(s) - x)| + h^{-2/m} |\partial_x^\alpha \partial_\xi^\beta \eta(s)| + 1) ds \end{aligned}$$

and Gronwall's inequality thus implies

$$|\partial_x^\alpha \partial_\xi^\beta (X(t) - x)| + h^{-2/m} |\partial_x^\alpha \partial_\xi^\beta (\eta(t) - h\partial_x A(x))| \lesssim h^{2/m} |t|.$$

On the other hand, since $\partial_\xi \dot{\eta} = -h^2 \partial_x^2 V(X) \partial_\xi X = O(h^{4/m})$ if $|t| \lesssim h^{-2/m}$, we have

$$|\partial_x^\alpha \partial_\xi^\beta (\eta(t) - \xi)| \lesssim h^{4/m} |t|.$$

Combining these with the fact that $h\partial_x^2 A(x) = O(h^{2/m} \langle x \rangle^{-1})$ on $\Gamma^h(L)$, we obtain

$$\begin{aligned} \partial_x \Xi &= \partial_x (\eta - hA(x)) - h\partial_x A(X) (\partial_x X - x) + h\partial_x A(x) - h\partial_x A(X) = O(h^{2/m} \langle x \rangle^{-1} |t|), \\ \partial_\xi (\Xi - \xi) &= \partial_\xi (\eta - \xi) - h\partial_x A(X) \partial_\xi X = O(h^{4/m} |t|). \end{aligned}$$

The estimates of higher order derivatives are verified by induction and we omit details. \square

3.2 Semiclassical paramatrix

We now turn into the construction of parametrices. We begin with the general case.

Theorem 3.4. *Let $L \geq 1$. Then, there exists $\delta > 0$ such that the following statements are satisfied uniformly with respect to $h \in (0, 1]$ and $1 \leq R \leq h^{-2/m}$:*

(1) *There exists $S^h \in C^\infty((-\delta R, \delta R) \times \mathbb{R}^{2d})$, which solves the Hamilton-Jacobi equation:*

$$\begin{cases} \partial_t S^h(t, x, \xi) + p^h(x, \partial_x S^h(t, x, \xi)) = 0, & (t, x, \xi) \in (-\delta R, \delta R) \times \Omega^h(R/3, 3L), \\ S^h(0, x, \xi) = x \cdot \xi, & (x, \xi) \in \Omega^h(R/3, 3L), \end{cases} \quad (3.13)$$

such that

$$|\partial_x^\alpha \partial_\xi^\beta (S^h(t, x, \xi) - x \cdot \xi + t\tilde{p}^h(x, \xi))| \leq C_{\alpha\beta} \langle x \rangle^{-1-\min(|\alpha|, 1)} |t|^2, \quad (3.14)$$

uniformly in $(t, x, \xi) \in (-\delta R, \delta R) \times \mathbb{R}^{2d}$, where \tilde{p}^h is given by (3.2) with L replaced $3L$.

(2) *For any $\chi^h \in S(1, g)$ supported in $\Omega^h(R, L)$ and integer $N \geq 0$, there exists a bounded family $\{a^h(t); |t| \leq \delta R, h \in (0, 1]\} \subset S(1, g)$ with $\text{supp } a^h(t) \subset \Omega^h(R/2, 2L)$ such that*

$$e^{-itP^h/h} \chi^h(x, hD) = J_{S^h}(a^h) + Q^h(t, N),$$

where $P^h = h^2 P$ and $J_{S^h}(a^h)$ is the h -FIO with the phase S^h and the amplitude a^h defined by

$$J_{S^h}(a^h)f(x) = (2\pi h)^{-d} \int e^{i(S^h(t, x, \xi) - y \cdot \xi)/h} a^h(t, x, \xi) f(y) dy d\xi,$$

and the remainder $Q^h(t, N)$ satisfies

$$\sup_{|t| \leq \delta R} \|Q^h(t, N)\|_{L^2 \rightarrow L^2} \leq C_N h^{N-1-2/m}. \quad (3.15)$$

Moreover, if we denote the kernel of $J_{S^h}(a^h)$ by $K^h(t, x, \xi)$ then

$$|K^h(t, x, y)| \lesssim \min\{h^{-d}, |th|^{-d/2}\}, \quad x, \xi \in \mathbb{R}^d, h \in (0, 1], |t| \leq \delta R. \quad (3.16)$$

Proof. Construction of the phase S^h : Let $X(t), \Xi(t)$ and $Y(t)$ be as in Lemma 3.2 with (R, L) replaced by $(R/4, 4L)$. Define an action integral \tilde{S}^h on $(-\delta R, \delta R) \times \Omega^h(R/4, 4L)$ by

$$\tilde{S}^h(t, x, \xi) := x \cdot \xi + \int_0^t L^h(X(s, Y(t, x, \xi), \xi), \Xi(s, Y(t, x, \xi), \xi)) ds,$$

where $L^h = \xi \cdot \partial_\xi p^h - p^h$ is the Lagrangian associated to p^h . A direct computation yields that \tilde{S}^h solves (3.13) and satisfies $(\partial_\xi \tilde{S}^h, \partial_x \tilde{S}^h) = (Y(t, x, \xi), \Xi(t, Y(t, x, \xi), \xi))$. Furthermore, the following conservation law holds:

$$p^h(x, \partial_x \tilde{S}^h(t, x, \xi)) = p^h(x, \Xi(t, Y(t, x, \xi), \xi)) = p^h(Y(t, x, \xi), \xi).$$

Here, by virtue of Lemma 3.2 (2), taking $\delta > 0$ smaller if necessary we learn that $h^2 \langle Y(t) \rangle^m \leq 5L$ on $(-\delta R, \delta R) \times \Omega^h(R/4, 4L)$ and hence p^h can be replaced by \tilde{p}^h since $\tilde{p}^h \equiv p^h$ on $\Gamma^h(6L)$. By Lemma 3.2 (2) and the fact that

$$|\partial_x \tilde{p}^h(x, \xi)| \lesssim \langle x \rangle^{-1} \text{ on } \Omega^h(R/4, 4L), \quad (3.17)$$

we have

$$\begin{aligned} |\tilde{p}^h(x, \partial_x \tilde{S}^h(t, x, \xi)) - \tilde{p}^h(x, \xi)| &\lesssim |Y(t) - x| \int_0^1 |(\partial_x \tilde{p}^h)(\lambda x + (1-\lambda)Y(t), \xi)| d\lambda \\ &\lesssim \langle x \rangle^{-1} |t| \end{aligned} \quad (3.18)$$

for $|t| \leq \delta R$ and $(x, \xi) \in \Omega^h(R/4, 4L)$, uniformly in h . We similarly obtain from Lemma 3.3 (2) and an induction on $|\alpha + \beta|$ that

$$|\partial_x^\alpha \partial_\xi^\beta (\tilde{p}^h(x, \partial_x \tilde{S}^h(t, x, \xi)) - \tilde{p}^h(x, \xi))| \leq C_{\alpha\beta} \langle x \rangle^{-1-\min\{|\alpha|, 1\}} |t|, \quad |\alpha + \beta| \geq 1. \quad (3.19)$$

Integrating with respect to t and using Hamilton-Jacobi equation (3.13), we learn that \tilde{S}^h satisfies (3.14) on $\Omega^h(R/4, 4L)$. Choosing $\psi \in S(1, g)$ so that $\text{supp } \psi \subset \Omega^h(R/4, 4L)$ and $\psi \equiv 1$ on $\Omega^h(R/3, 3L)$, we extend \tilde{S}^h to the whole space \mathbb{R}^{2d} as follows:

$$S^h(t, x, \xi) = x \cdot \xi - t \tilde{p}^h(x, \xi) + \psi(x, \xi) \left(\tilde{S}^h(t, x, \xi) - x \cdot \xi + t \tilde{p}^h(x, \xi) \right).$$

$S^h(t, x, \xi)$ clearly obeys the assertions in (1).

Construction of the amplitude a^h : Let us make the following ansatz:

$$v(t, x) = \frac{1}{(2\pi h)^d} \int e^{i(S^h(t, x, \xi) - y \cdot \xi)/h} a^h(t, x, \xi) f(y) dy d\xi,$$

where $a^h = \sum_{j=0}^{N-1} h^j a_j^h$. In order to approximately solve the Schrödinger equation

$$(hD_t + P^h)v(t) = O(h^N); \quad v|_{t=0} = \chi^h(x, hD)u_0,$$

the amplitude should satisfy the following transport equations:

$$\begin{cases} \partial_t a_0^h + \mathcal{X}^h \cdot \partial_x a_0^h + \mathcal{Y}^h a_0^h = 0; & a_0^h|_{t=0} = \chi^h, \\ \partial_t a_j^h + \mathcal{X}^h \cdot \partial_x a_j^h + \mathcal{Y}^h a_j^h + iK a_{j-1}^h = 0; & a_j^h|_{t=0} = 0, \quad 1 \leq j \leq N-1, \end{cases} \quad (3.20)$$

where $K = -\frac{1}{2} \partial_j g^{jk}(x) \partial_k$, a vector field \mathcal{X} and a function \mathcal{Y} are defined by

$$\mathcal{X}(t, x, \xi) := (\partial_\xi p^h)(x, \partial_x S^h(t, x, \xi)), \quad \mathcal{Y}(t, x, \xi) := [k(x, \partial_x) S^h + p_1^h(x, \partial_x S^h)](t, x, \xi).$$

Note that (3.14) and (1.10) imply

$$|\partial_x^\alpha \partial_\xi^\beta \mathcal{Y}(t, x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1} \quad \text{on} \quad (-\delta R, \delta R) \times \Omega(R/3, 3L). \quad (3.21)$$

The system (3.20) can be solved by the standard method of characteristics along the flow generated by $\mathcal{X}(t, x, \xi)$. More precisely, let us consider the following ODE

$$\partial_t z(t, s, x, \xi) = \mathcal{X}(t, z(t, s, x, \xi), \xi); \quad z(s, s) = x.$$

Then, by virtue of (3.18) and (3.19), the same argument as that in Subsection 3.1 yields that there exists $\delta > 0$ such that, for any fixed $h \in (0, 1]$, $1 \leq R \leq h^{-2/m}$, $z(t, s, x, \xi)$ is well-defined for $t, s \in (-\delta R, \delta R)$ and $(x, \xi) \in \Omega(R/3, 3L)$, and satisfies

$$|z(t, s) - x| \leq C|t - s|, \quad |\partial_x^\alpha \partial_\xi^\beta (z(t, s) - x)| \leq C_{\alpha\beta} \langle x \rangle^{-1} |t - s|, \quad |\alpha + \beta| \geq 1. \quad (3.22)$$

For $(t, x, \xi) \in (-\delta R, \delta R) \times \Omega(R/3, 3L)$, we then define a_j , $j = 0, 1, \dots, N-1$, inductively by

$$\begin{aligned} a_0(t, x, \xi) &= \chi^h(z(0, t, x, \xi), \xi) \exp \left(\int_0^t \mathcal{Y}(s, z(s, t, x, \xi), \xi) ds \right), \\ a_j(t, x, \xi) &= - \int_0^t (iK a_{j-1})(s, z(s, t, x, \xi), \xi) \exp \left(\int_u^t \mathcal{Y}(u, z(u, t, x, \xi), \xi) du \right) ds. \end{aligned}$$

It is easy to see from (3.22) and $\text{supp } \chi^h \subset \Omega^h(R, L)$ that $\text{supp } a_j \subset \Omega^h(R/2, 2L)$ for all $|t| \leq \delta R$. Furthermore, taking $\delta > 0$ smaller if necessary we see that a_j are smooth on $\Omega(5R/12, 12L/5)$.

Since $\Omega^h(R/2, 2L) \Subset \Omega(5R/12, 12L/5) \Subset \Omega(R/3, 3L)$, if we extend a_j to the whole space \mathbb{R}^{2d} so that $a_j \equiv 0$ outside $\Omega^h(R/2, 2L)$, then a_j are still smooth. We further learn by (3.22), (3.21) and the fact $\chi \in S(1, g)$ that $a_j \in S(1, g)$ uniformly with respect to $|t| \leq \delta R$ and $h \in (0, 1]$. Finally, one can check by a direct computation that a_j solve the system (3.20).

Justification of the parametrix: At first note that, since $|\partial_\xi \otimes \partial_x p^h(x, \xi)| \lesssim \langle x \rangle^{-1}$ on $\Gamma^h(3L)$, if $\delta > 0$ is small enough then (3.14) implies

$$|\partial_\xi \otimes \partial_x S^h(t, x, \xi) - \text{Id}| < 1/2 \quad \text{for } (t, x, \xi) \in (-\delta R, \delta R) \times \Omega^h(R/3, 3L).$$

The standard h -FIO theory then shows that, for any amplitude $b^h \in S(1, g)$ supported in $\Omega^h(R/2, 2L)$ ($\Subset \Omega^h(R/3, 3L)$), the associated h -FIO $J_{S^h}(b^h)$ is uniformly bounded on L^2 with respect to h :

$$\sup_{|t| \leq \delta R} \|J_{S^h}(b^h)\|_{L^2 \rightarrow L^2} \lesssim 1, \quad h \in (0, 1], \quad 1 \leq R \leq h^{-2/m}.$$

We now prove the remainder estimate (3.15). We may assume $t \geq 0$ without loss of generality, and the proof for the opposite case being analogous. By the Duhamel formula, we have

$$\begin{aligned} e^{-itP^h/h} \chi^h(x, hD) &= J_{S^h}(a^h) + Q^h(t, N), \\ Q^h(t, N) &= -\frac{i}{h} \int_0^t e^{-i(t-s)P^h/h} (hD_t + P^h) J_{S^h}(a^h)|_{t=s} ds. \end{aligned}$$

By (3.13), (3.20) and direct computations, we obtain $(hD_t + P^h)J_{S^h}(a^h) = -ih^N J_{S^h}(Ka_{N-1}^h)$. Since $\text{supp } Ka_{N-1}^h \subset \Omega(R/2, 2L)$ and $Ka_{N-1}^h \in S(1, g)$, $J_{S^h}(P^h a_{N-1}^h)$ is bounded on L^2 uniformly in $h \in (0, 1]$, $1 \leq R \leq h^{-2/m}$ and $0 \leq t \leq \delta R$, and (3.15) follows.

Dispersive estimates: The kernel of $J_{S^h}(a^h)$ is given by

$$K^h(t, x, y) = (2\pi h)^{-d} \int e^{i(S^h(t, x, \xi) - y \cdot \xi)/h} a^h(t, x, \xi) d\xi.$$

If $|t| \leq h$, then the assertion is obvious since a^h is compactly supported in ξ . On the other hand, by virtue of (3.14), we have

$$\frac{\partial_\xi^2 S^h(t, x, \xi)}{t} = -(g^{jk}(x))_{j,k} + O(\delta), \quad h \leq |t| \leq \delta R,$$

and $|t^{-1} \partial_x^\alpha \partial_\xi^\beta S^h(t, x, \xi)| \leq C_{\alpha\beta}$ if $h \leq |t| \leq \delta R$ and $|\alpha + \beta| \geq 2$. As a consequence, since $g^{jk}(x)$ is uniformly elliptic, the phase function $t^{-1}(S^h(t, x, \xi) - y \cdot \xi)$ has a unique non-degenerate critical point for all $h \leq |t| \leq \delta R$ and we can apply the stationary phase method to $K^h(t, x, y)$, provided that $\delta > 0$ is small enough. Therefore,

$$|K^h(t, x, y)| \lesssim h^{-d} |th^{-1}|^{-d/2} \lesssim |th|^{-d/2}, \quad h \leq |t| \leq \delta R, \quad x, \xi \in \mathbb{R}^d, \quad h \in (0, 1],$$

from which we complete the proof. \square

We next state the flat case.

Theorem 3.5 (Flat case). *Suppose that $g^{jk} \equiv \delta_{jk}$ and $L \geq 1$. Then, there exists $\delta > 0$ such that the following statements are satisfied:*

(1) *There exists $S^h \in C^\infty((-\delta h^{-2/m}, \delta h^{-2/m}) \times \mathbb{R}^{2d})$, which solves the Hamilton-Jacobi equation:*

$$\begin{cases} \partial_t S^h(t, x, \xi) + p^h(x, \partial_x S^h(t, x, \xi)) = 0, & (t, x, \xi) \in (-\delta h^{-2/m}, \delta h^{-2/m}) \times \Gamma^h(3L), \\ S^h(0, x, \xi) = x \cdot \xi, & (x, \xi) \in \Gamma^h(3L), \end{cases}$$

such that

$$|\partial_x^\alpha \partial_\xi^\beta (S^h(t, x, \xi) - x \cdot \xi + t\tilde{p}^h(x, \xi))| \leq C_{\alpha\beta} h^{2(1+\min\{|\alpha|, 1\})/m} |t|^2$$

uniformly in $(t, x, \xi) \in (-\delta h^{-2/m}, \delta h^{-2/m}) \times \mathbb{R}^{2d}$ and $h \in (0, 1]$.

(2) For any $\chi^h \in S(1, g)$ supported in $\Gamma^h(L)$ and integer $N \geq 0$, there exists a bounded family $\{a^h(t); t \in (-\delta h^{-2/m}, \delta h^{-2/m}), h \in (0, 1]\} \subset S(1, g)$ with $\text{supp } a^h(t) \subset \Gamma^h(2L)$ such that

$$e^{-itH^h/h} \chi^h(x, hD) = J_{S^h}(a^h) + Q^h(t, N),$$

where $J_{S^h}(a^h)$ is the h -FIO with the phase S^h and the amplitude a^h and the remainder $Q^h(t, N)$ satisfies

$$\sup_{|t| \leq \delta h^{-2/m}} \|Q^h(t, N)\|_{L^2 \rightarrow L^2} \leq C_N h^{N-1-2/m}, \quad h \in (0, 1].$$

Finally, the kernel of $J_{S^h}(a^h)$ satisfies dispersive estimates (3.16) for $|t| \leq \delta h^{-2/m}$.

The proof is almost analogous and the only difference is to use Lemma 3.3 and the fact that $\partial_x \tilde{p}^h = O(h^{2/m})$ on $\Gamma^h(4L)$ instead of Lemma 3.2 and (3.17), respectively. We hence omit the details.

4 Proof of main theorems

In this section we prove Theorems 1.3 and 1.4. The general strategy is basically same as that of [22, Section 7]. Recall that the energy shell $\Gamma^h(L)$ is defined by

$$\Gamma^h(L) = \{(x, \xi) \in \mathbb{R}^{2d}; |\xi|^2 + h^2 \langle x \rangle^m < L\}.$$

We begin with the following theorem which is a consequence of Theorem 3.5.

Theorem 4.1. *Fix $L > 0$. Then, for sufficiently small $\delta > 0$ depending only on L , the following statements are satisfied:*

(1) For any $h \in (0, 1]$, $1 \leq R \leq h^{-2/m}$ and symbol $\chi_R^h \in S(1, g)$ supported in $\{|x| > R\} \cap \Gamma^h(L)$,

$$\|\chi_R^h(x, hD) e^{-itP} \chi_R^h(x, hD)^*\|_{L^1 \rightarrow L^\infty} \leq C_\delta |t|^{-d/2}, \quad 0 < |t| < \delta h R, \quad (4.1)$$

where $C_\delta > 0$ may be taken uniformly with respect to h and R .

(2) For $h \in (0, 1]$ and any symbol $\chi^h \in S(1, g)$ supported in $\Gamma^h(L)$,

$$\|\chi^h(x, hD) e^{-itP} \chi^h(x, hD)^*\|_{L^1 \rightarrow L^\infty} \leq C_\delta |t|^{-d/2}, \quad 0 < |t| < \delta h. \quad (4.2)$$

Proof. The expansion formula (2.3) and Lemma 2.1 shows that there exists $\chi_{0,R}^h, \chi_1^h \in S(1, g)$ with $\chi_{0,R}^h \subset \text{supp } \chi_R^h$ such that $\chi_R^h(x, hD)^* = \chi_{0,R}^h(x, hD) \chi_1^h(x, hD) + O_{L^p \rightarrow L^q}(h^N)$ for any $1 \leq p \leq q \leq \infty$ and any $N \geq 0$. We hence can replace $\chi_R^h(x, hD)^*$ by $\chi_{0,R}^h(x, hD) \chi_1^h(x, hD)$ without loss of generality. Then, the assertion follows from (3.15), (3.16) and (2.1). \square

Using Theorem 4.1, Keel-Tao's abstract theorem and the Duhamel formula, one can obtain the following semiclassical Strichartz estimates with an inhomogeneous term. The proof is same as that of [23, Proposition 7.4] (see also [2, Section 5]), and we omit it.

Proposition 4.2. *Let $T > 0$ and (p, q) satisfy (1.4). Under conditions in Theorem 4.1, we have*

$$\begin{aligned} \|\chi_R^h(x, hD)e^{-itP}u_0\|_{L_T^p L^q} &\lesssim h\|u_0\|_{L^2} + \|\chi_R^h(x, hD)u_0\|_{L^2} \\ &\quad + (hR)^{-1/2}\|\chi_R^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^2} \\ &\quad + (hR)^{1/2}\|[H, \chi_R^h(x, hD)]e^{-itP}u_0\|_{L_T^2 L^2}, \end{aligned} \quad (4.3)$$

$$\begin{aligned} \|\chi^h(x, hD)e^{-itP}u_0\|_{L_T^p L^q} &\lesssim h\|u_0\|_{L^2} + \|\chi^h(x, hD)u_0\|_{L^2} \\ &\quad + h^{-1/2}\|\chi^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^2} \\ &\quad + h^{1/2}\|[H, \chi^h(x, hD)]e^{-itP}u_0\|_{L_T^2 L^2}, \end{aligned} \quad (4.4)$$

uniformly with respect to $h \in (0, 1]$ and $1 \leq R \leq h^{-2/m}$, where implicit constants depend on T .

Proof of Theorem 1.3. Recall that Proposition 2.2, together with Minkowski's inequality, shows

$$\|e^{-itP}u_0\|_{L_T^p L^q} \lesssim \|u_0\|_{L^2} + \sum_{k=0,1} \left(\sum_h \|\Psi_k^h(x, hD)e^{-itP}u_0\|_{L_T^p L^q}^2 \right)^{1/2},$$

with $\Psi_k^h \in S(1, h^{4/m}dx^2 + d\xi^2/\langle \xi \rangle^2)$ satisfying $\text{supp } \Psi_0^h \subset \{\langle x \rangle \lesssim h^{-2/m}, |\xi| \approx 1\}$ and $\text{supp } \Psi_1^h \subset \{\langle x \rangle \approx h^{-2/m}, |\xi| \lesssim 1\}$.

We first study $\Psi_1^h(x, hD)e^{-itP}$. The expansion formula (2.2) shows

$$\text{supp Sym}([P, \Psi_1^h(x, hD)]) \subset \text{supp } \Psi_1^h, \quad \text{Sym}([P, \Psi_1^h(x, hD)]) \in S(h^{-1+2/m}, g).$$

Therefore, using Lemma 2.1 we have

$$\|[P, \Psi_1^h(x, hD)]e^{-itP}u_0\|_{L_T^2 L^2} \lesssim h^{-1/2+1/m}\|\tilde{\Psi}_1^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^2} + h\|u_0\|_{L^2}, \quad (4.5)$$

where $\tilde{\Psi}_1^h \in S(1, g)$ is of the form $\tilde{\Psi}_1^h(x, \xi) = \tilde{\theta}(h^{2/m}x)\tilde{\psi}_1(x, \xi/h)$ with $\tilde{\theta} \in C_0^\infty(\mathbb{R}^d)$ supported in $\{|x| \approx 1\}$ and with $\tilde{\psi}_1 \in S(1, g)$ supported in $\{|\xi|^2 \lesssim \langle x \rangle^m\}$. In particular, $\tilde{\Psi}_1^h \equiv 1$ on $\text{supp } \Psi_1^h$. Applying Proposition 4.2 to $\Psi_1^h(x, hD)e^{-itP}$ with $R \approx h^{-2/m}$ and using (4.5), we then obtain

$$\begin{aligned} \|\Psi_1^h(x, hD)e^{-itP}u_0\|_{L_T^p L^q} &\lesssim h\|u_0\|_{L^2} + \|\Psi_1^h(x, hD)u_0\|_{L^2} \\ &\quad + h^{-1/2+1/m}\|\Psi_1^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^2} + h^{1/2-1/m}\|[P, \Psi_1^h(x, hD)]e^{-itP}u_0\|_{L_T^2 L^2} \\ &\lesssim h\|u_0\|_{L^2} + \|\Psi_1^h(x, hD)u_0\|_{L^2} + h^{-1/2+1/m}\|\tilde{\Psi}_1^h(x, hD)e^{-itP}u_0\|_{L_T^2 L^2} \\ &\lesssim h\|u_0\|_{L^2} + \|\theta(h^{2/m}x)\psi_1(x, D)u_0\|_{L^2} + \|\tilde{\theta}(h^{2/m}x)\langle x \rangle^{m/4-1/2}\tilde{\psi}_1(x, D)e^{-itP}u_0\|_{L_T^2 L^2}, \end{aligned}$$

where, in the last line, we have used the fact that $h^{-1/2+1/m} \approx \langle x \rangle^{m/4-1/2}$ on $\text{supp } \tilde{\Psi}_1^h$. Combining this estimate with the following the norm equivalence:

$$\|v\|_{L^2}^2 \approx \sum_h \|\theta(h^{2/m}x)v\|_{L^2}^2 \approx \sum_h \|\tilde{\theta}(h^{2/m}x)v\|_{L^2}^2$$

(which follows from the almost orthogonality of $\theta(h^{2/m}x)$ and $\tilde{\theta}(h^{2/m}x)$), we have

$$\sum_h \|\Psi_1^h(x, hD)e^{-itP}u_0\|_{L_T^p L^q}^2 \lesssim \|u_0\|_{L^2}^2 + \|\langle x \rangle^{m/4-1/2}\tilde{\psi}_1(x, D)e^{-itP}u_0\|_{L_T^2 L^2}^2.$$

On the other hand, since $e_s \in S((1 + |\xi|^2 + |x|^m)^{s/2}, g)$, we see that $\langle x \rangle^{m/4-1/2} \tilde{\psi}_1 e_{-1/2+1/m} \in S(1, g)$ and hence $\langle x \rangle^{m/4-1/2} \tilde{\psi}_1(x, D) E_{1/2-1/m}^{-1}$ is bounded on L^2 . Applying Lemma 2.7 and using the unitarity of e^{-itP} we conclude

$$\begin{aligned} \sum_h \|\Psi_1^h(x, hD) e^{-itP} u_0\|_{L_T^p L^q}^2 &\lesssim \|u_0\|_{L^2}^2 + \|E_{1/2-1/m} e^{-itP} u_0\|_{L_T^2 L^2}^2 \\ &\lesssim \|E_{1/2-1/m} u_0\|_{L^2}^2. \end{aligned} \quad (4.6)$$

Next we study $\Psi_0^h(x, hD) e^{-itP} u_0$. Consider a dyadic partition of unity on $\pi_x(\text{supp } \Psi_0^h)$:

$$\varphi_{-1}(x) + \sum_{0 \leq j \leq j_h} \varphi(2^{-j}x) = 1, \quad x \in \pi_x(\text{supp } \Psi_0^h),$$

where $j_h \lesssim (2/m) \log(1/h)$ and $\varphi_{-1}, \varphi \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp } \varphi_{-1} \subset \{|x| < 1\}$ and $\text{supp } \varphi \subset \{1/2 < |x| < 2\}$. We set $\varphi_j(x) = \varphi(2^{-j}x)$ for $j \geq 0$. Since $p, q \geq 2$, it follows from Minkowski's inequality that

$$\|\Psi_0^h(x, hD) e^{-itP} u_0\|_{L_T^p L^q}^2 \leq \sum_{-1 \leq j \leq j_h} \|\varphi_j(x) \Psi_0^h(x, hD) e^{-itP} u_0\|_{L_T^p L^q}^2.$$

We here choose cut-off functions $\tilde{\varphi}_{-1}, \tilde{\varphi} \in C_0^\infty(\mathbb{R}^d)$ and $\tilde{\Psi}_0^h \in S(1, g)$ supported in a small neighborhood of $\text{supp } \varphi_{-1}$, $\text{supp } \varphi$ and $\text{supp } \Psi_0^h$, respectively, so that $\tilde{\varphi}_{-1} \equiv 1$ on $\text{supp } \varphi_{-1}$, $\tilde{\varphi} \equiv 1$ on $\text{supp } \varphi$ and $\tilde{\Psi}_0^h \equiv 1$ on $\text{supp } \Psi_0^h$. Set $\tilde{\varphi}_j(x) = \tilde{\varphi}(2^{-j}x)$ for $j \geq 0$.

$$\text{supp } \tilde{\varphi}_j \tilde{\Psi}_0^h \subset \{|x| \approx 2^j, |\xi| \approx 1\}, \quad \tilde{\varphi}_j \tilde{\Psi}_0^h \equiv 1 \text{ on } \text{supp } \varphi_j \Psi_0^h.$$

Since the symbolic calculus shows

$$\text{supp Sym}([P, \varphi_j(x) \Psi_0^h(x, hD)]) \subset \text{supp}(\varphi_j \Psi_0^h), \quad \text{Sym}([P, \varphi_j(x) \Psi_0^h(x, hD)]) \in S(2^{-j}h^{-1}, g),$$

applying Proposition 4.2 with $R = 2^j$, we learn by a similar argument as above that

$$\begin{aligned} &\|\varphi_j(x) \Psi_0^h(x, hD) e^{-itP} u_0\|_{L_T^p L^q} \\ &\lesssim h \|u_0\|_{L^2} + \|\varphi_j(x) \Psi_0^h(x, hD) u_0\|_{L^2} + (h2^j)^{-1/2} \|\tilde{\varphi}_j(x) \tilde{\Psi}_0^h(x, hD) e^{-itP} u_0\|_{L_T^2 L^2} \\ &\lesssim h \|u_0\|_{L^2} + \|\varphi_j(x) \Psi_0^h(x, hD) u_0\|_{L^2} + \|\tilde{\varphi}_j(x) \langle x \rangle^{-1/2} \langle D \rangle^{1/2} \tilde{\Psi}_0^h(x, hD) e^{-itP} u_0\|_{L_T^2 L^2}. \end{aligned}$$

The almost orthogonality of φ_j and $\tilde{\varphi}_j$ then yields

$$\begin{aligned} &\sum_{-1 \leq j \leq j_h} \|\varphi_j(x) \Psi_0^h(x, hD) e^{-itP} u_0\|_{L_T^p L^q}^2 \\ &\lesssim (h \log(1/h))^2 \|u_0\|_{L^2}^2 + \|\Psi_0^h(x, hD) u_0\|_{L^2}^2 + \|\langle x \rangle^{-1/2} \langle D \rangle^{1/2} \tilde{\Psi}_0^h(x, hD) e^{-itP} u_0\|_{L_T^2 L^2}^2. \end{aligned}$$

We here claim that, for any $\nu \geq 0$,

$$\begin{aligned} &\|\langle x \rangle^{-1/2} \langle D \rangle^{1/2} \tilde{\Psi}_0^h(x, hD) e^{-itP} u_0\|_{L_T^2 L^2} \\ &\lesssim \|\tilde{\Psi}_0^h(x, hD) \langle x \rangle^{-1/2-m\nu/2} E_{1/2+\nu}^{-1} e^{-itP} u_0\|_{L_T^2 L^2} + h^{1/2-\nu} \|u_0\|_{L^2}. \end{aligned}$$

Indeed, this estimate can be proved by using following two bounds:

$$\begin{aligned} &\langle x \rangle^{-1/2} \langle D \rangle^{1/2} \langle x \rangle^{1/2+m\nu/2} E_{1/2+\nu}^{-1} = O_{L^2 \rightarrow L^2}(1), \\ &[\langle x \rangle^{-1/2-m\nu/2} E_{1/2+\nu}, \tilde{\Psi}_0^h(x, hD)] = O_{L^2 \rightarrow L^2}(h^{1/2-\nu}). \end{aligned}$$

Note that these bounds follow from the following computations

$$\begin{aligned}\langle x \rangle^{-1/2} \langle \xi \rangle^{1/2} &\lesssim \langle x \rangle^{-1/2-m\nu/2} e_{1/2+\nu}(x, \xi) \text{ for any } \nu \geq 0, \\ \{e_{1/2+\nu}, \tilde{\Psi}_0^h(\cdot, h\cdot)\} &= O(\langle x \rangle^{-1} h e_{1/2+\nu}) = O(\langle x \rangle^{-1} h^{1/2-\nu}).\end{aligned}$$

We now choose a cut-off $\tilde{\theta} \in C_0^\infty(\mathbb{R}^d)$ supported away from the origin such that $\tilde{\theta} \equiv 1$ on $\pi_\xi(\text{supp } \Psi_0^h)$. Lemma 2.1 then yields

$$\|\Psi_0^h(x, hD)(1 - \tilde{\theta}(hD))\|_{L^2 \rightarrow L^q} + \|\tilde{\Psi}_0^h(x, hD)(1 - \tilde{\theta}(hD))\|_{L^2 \rightarrow L^q} \leq Ch, \quad 2 \leq q \leq \infty, \quad h \in (0, 1],$$

and we thus may replace $\Psi_0^h(x, hD)$ and $\tilde{\Psi}_0^h(x, hD)$ by $\Psi_0^h(x, hD)\tilde{\theta}(hD)$ and $\tilde{\Psi}_0^h(x, hD)\tilde{\theta}(hD)$, respectively. Then, by the almost orthogonality of $\tilde{\theta}(h\xi)$, we obtain for $0 \leq \nu < 1/2$,

$$\begin{aligned}\sum_h h^{1-2\nu} \|u_0\|_{L^2}^2 + \|\Psi_0^h(x, hD)\tilde{\theta}(hD)u_0\|_{L^2}^2 &\lesssim \|u_0\|_{L^2}^2, \\ \sum_h \|\tilde{\Psi}_0^h(x, hD)\tilde{\theta}(hD)\langle x \rangle^{-1/2-m\nu/2} E_{1/2+\nu} e^{-itP} u_0\|_{L_T^2 L^2}^2 &\lesssim \|\langle x \rangle^{-1/2-m\nu/2} E_{1/2+\nu} e^{-itP} u_0\|_{L_T^2 L^2}^2.\end{aligned}$$

We now apply Proposition 2.8 with $s = 1/2 - 1/m + \nu$ and conclude

$$\sum_h \|\Psi_0^h(x, hD) e^{-itP} u_0\|_{L_T^p L^q}^2 \leq C_{T,\nu} \|E_{1/2-1/m+\nu} u_0\|_{L^2}^2, \quad T > 0, \quad 0 < \nu < 1/2. \quad (4.7)$$

Summing the estimates (4.6) and (4.7) and using $e_s \lesssim \langle \xi \rangle^s + \langle x \rangle^{ms/2}$, we conclude

$$\|e^{-itP} u_0\|_{L_T^p L^q} \leq C_{T,\nu} \|\langle D \rangle^{1/2-1/m+\nu} u_0\|_{L^2} + C_{T,\nu} \|\langle x \rangle^{m/4-1/2+\nu} u_0\|_{L^2}, \quad \nu > 0,$$

for any admissible pair (p, q) with $q < \infty$. Finally, if $d \geq 3$, then Theorem 1.3 can be verified by interpolation the $L_T^2 L^{2d/(d-2)}$ -estimate with the trivial $L_T^\infty L^2$ -estimate. For $d = 2$, let us fix $\varepsilon > 0$ and an admissible pair (p, q) arbitrarily and choose an admissible pair (p_0, q_0) with $2 < p_0 < p$ and $\nu > 0$ so that

$$\left(\frac{1}{2} - \frac{1}{m} + \nu\right) \frac{p_0}{p} = \left(\frac{1}{2} - \frac{1}{m}\right) \frac{2}{p} + \varepsilon.$$

Interpolating the $L_T^{p_0} L^{q_0}$ -estimate with the $L_T^\infty L^2$ -estimate, we have

$$\|e^{-itP} u_0\|_{L_T^p L^q} \leq C_{T,\nu} \|\langle D \rangle^{(1-1/m)/p+\varepsilon} u_0\|_{L^2} + C_{T,\nu} \|\langle x \rangle^{(m/2-1)/p+\varepsilon} u_0\|_{L^2}.$$

Finally, we refer to *e.g.*, [32] for the interpolation in weighted spaces. \square

Next we prove Theorem 1.4. Hence, in what follows (in this section), we suppose that $H = \frac{1}{2}(D - A(x))^2 + V(x)$ satisfies Assumption A. In this case, we first obtain a slightly long-time dispersive estimate which is better than Theorem 4.1 (2).

Theorem 4.3. *Let $I \Subset (0, \infty)$ be an interval and $\delta > 0$ small enough. Then, for any $h \in (0, 1]$ and symbol $\chi^h \in S(1, g)$ supported in $\Gamma^h(L)$,*

$$\|\chi^h(x, hD) e^{-itH} \chi^h(x, hD)^*\|_{L^1 \rightarrow L^\infty} \leq C_\delta |t|^{-d/2}, \quad 0 < |t| < \delta h^{1-2/m}.$$

As in the previous argument, we then have the following:

Proposition 4.4. *Under conditions in Theorem 4.3, we have*

$$\begin{aligned} \|\chi^h(x, hD)e^{-itH}u_0\|_{L_T^p L^q} &\lesssim h\|u_0\|_{L^2} + \|\chi^h(x, hD)u_0\|_{L^2} \\ &\quad + h^{-1/2+1/m}\|\chi^h(x, hD)e^{-itH}u_0\|_{L_T^2 L^2} \\ &\quad + h^{1/2-1/m}\|[H, \chi^h(x, hD)]e^{-itH}u_0\|_{L_T^2 L^2}, \end{aligned}$$

uniformly in h .

Proof of Theorem 1.4. The proof is analogous to that of Theorem 1.3. The only difference compared to the variable coefficient case is the following fact:

$$\text{Sym}([H, \Psi_0]) = h^{-2} \text{Sym}([H^h, \Psi_0^h]) \in S(h^{-1+2/m}, g). \quad (4.8)$$

(Recall that, in general case, we only have $\text{Sym}([P, \Psi_0^h]) \in S(h^{-1}\langle x \rangle^{-1}, g)$.) Indeed, since $\Psi_0^h \in S(1, h^{4/m}dx^2 + d\xi^2)$ and $\langle x \rangle^{m/2} \lesssim h^{-1}$ on $\text{supp } \Psi_0^h$, we have

$$\begin{aligned} \{(\xi - hA)^2, \Psi_0^h\} &= 2(\xi - hA) \cdot \partial_x \Psi_0^h - 2h^2 \partial_x A(\xi - A) \cdot \partial_\xi \Psi_0^h \\ &= O(h^{2/m} + h^{1+2/m}\langle x \rangle^{m/2} + h^2\langle x \rangle^{m-1}) = O(h^{2/m}). \end{aligned}$$

We similarly obtain $\{p_1^h, \Psi_0^h\} = O(h^{2/m}\langle x \rangle^{-1})$ and $\{h^2V, \Psi_0^h\} = O(h^{2/m})$. Therefore

$$\text{Sym}([H^h, \Psi_0^h]) = \frac{h}{i} \{(\xi - hA)^2/2 + hp_1^h + h^2V, \Psi_0^h\} + O(h^2) = O(h^{1+2/m}) \text{ in } S(1, g).$$

Applying Proposition 4.4 to $\Psi_0^h(x, hD)e^{-itH}u_0$ and using (4.8), we learn by the same argument as in the variable coefficient case that

$$\begin{aligned} \|\Psi_0^h(x, hD)e^{-itH}u_0\|_{L_T^2 L^{\frac{2d}{d-2}}} &\lesssim \|\Psi_0^h(x, hD)u_0\|_{L^2} + h\|u_0\|_{L^2} + \|\tilde{\Psi}_0^h(x, hD)E_{1/2-1/m}e^{-itH}u_0\|_{L_T^2 L^2}. \end{aligned}$$

By the almost orthogonality of the ξ -support of $\Psi_0^h(x, h\xi)$ and Lemma 2.7, we then conclude

$$\sum_h \|\Psi_0^h(x, hD)e^{-itH}u_0\|_{L_T^2 L^{\frac{2d}{d-2}}}^2 \lesssim \|u_0\|_{L^2}^2 + \|E_{1/2-1/m}e^{-itH}u_0\|_{L_T^2 L^2}^2 \lesssim \|E_{1/2-1/m}u_0\|_{L^2}^2.$$

which, together with the estimates (4.6) and Lemma 2.2, implies

$$\|e^{-itH}u_0\|_{L_T^2 L^{\frac{2d}{d-2}}} \leq C_T \|E_{1/2-1/m}u_0\|_{L^2}.$$

Finally, the assertion follows from an interpolation with the $L_T^\infty L^2$ -estimate. \square

A The case with growing potentials

We here prove Corollary 1.5 with a simpler proof than that of Theorem 1.4. Throughout this appendix we assume that $H = \frac{1}{2}(D - A(x))^2 + V(x)$ satisfies Assumption A and (1.5). The main point is the following square function estimates:

Proposition A.1. *Consider a 4-adic partition of unity on $[0, \infty)$:*

$$f_0, f \in C_0^\infty(\mathbb{R}), \text{ supp } f \subset [1/4, 4], \ 0 \leq f_0, f \leq 1, \ f_0(\lambda) + \sum_h f(h^2\lambda) = 1, \ \lambda \geq 0.$$

Then, for any $1 < q < \infty$, we have the square function estimates:

$$\|v\|_{L^q} \approx \left\| \left(|f_0(H)v|^2 + \sum_h f(h^2 H)v|^2 \right)^{1/2} \right\|_{L^q}. \quad (\text{A.1})$$

In particular, if $2 \leq q < \infty$ then

$$\|v\|_{L^q} \lesssim \left(\|f_0(H)v\|_{L^q}^2 + \sum_h \|f(h^2 H)v\|_{L^q}^2 \right)^{1/2}. \quad (\text{A.2})$$

Here, implicit constants are independent of v .

Proof. Since $V \geq 0$ and $V \in L^1_{loc}$, the heat kernel of the Schrödinger semigroup e^{-tH} satisfies the upper Gaussian bound (cf. Simon [29]):

$$|\partial_x^j e^{-tH}(x, y)| \leq C_d t^{-(d+j)/2} e^{-c|x-y|^2/t}, \quad t > 0, \quad j = 0, 1.$$

Then, a general theorem by Zheng [39] implies the square function estimates (A.1). (A.2) is an immediate consequence of (A.1) and Minkowski's inequality since $q \geq 2$. \square

Furthermore, thanks to the positivity of V , one can obtain an approximation theorem of the spectral multiplier $f(h^2 H)$ in terms of h - Ψ DO:

Proposition A.2. (1) Let $f \in C_0^\infty(\mathbb{R})$ with $\text{supp } f \Subset (0, \infty)$. Then, for any $N \geq 0$, there exists a bounded family $\{\chi^h\}_{h \in (0,1]} \subset S(1, g)$ with $\text{supp } \chi^h \subset \text{supp}(f \circ p^h)$ such that

$$\|f(h^2 H) - \chi^h(x, hD)\|_{L^2 \rightarrow L^q} \leq C_{qN} h^{N-d(1/2-1/q)}, \quad 2 \leq q \leq \infty,$$

uniformly in $h \in (0, 1]$. In particular, $f(h^2 H)$ is bounded from L^2 to L^q with the bounds:

$$\|f(h^2 H)\|_{L^2 \rightarrow L^q} \leq C_q h^{-d(1/2-1/q)}, \quad 2 \leq q \leq \infty, \quad h \in (0, 1].$$

(2) Let $f_0 \in C_0^\infty(\mathbb{R})$. Then, $\|f_0(H)\|_{L^2 \rightarrow W^{s,q}} \leq C_{qs}$ for any $2 \leq q \leq \infty$ and $s \geq 0$.

Proof. The proof is essentially same as in the case when $A \equiv V \equiv 0$ (see, e.g., [2]). Thus we only outline the proof. The important point is the following ellipticity:

$$p^h(x, \xi) \approx |\xi|^2 + h^2 \langle x \rangle^m, \quad (\text{A.3})$$

where the implicit constants are independent of $h \in (0, 1]$, which can be verified as follows: if $|\xi|^2 \geq Ch^2 \langle x \rangle^m$ for sufficiently large $C > 0$ then $p^h \geq |\xi|^2 - h^2 |A|^2 + h^2 V \gtrsim |\xi|^2 + h^2 \langle x \rangle^m$; otherwise, by Assumption A (1), $p^h \geq h^2 V \gtrsim |\xi|^2 + h^2 \langle x \rangle^m$ since $|\xi|^2 \lesssim h^2 \langle x \rangle^m$. The upper bound is obvious. Using this bound and (1.10), we see that

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(\frac{1}{p^h(x, \xi) - z} \right) \right| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|} |\text{Im } z|^{-1-|\alpha+\beta|},$$

uniformly in $x, \xi \in \mathbb{R}^d$ and $h \in (0, 1]$, and locally uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$. Then we can follow the standard argument (see, e.g., [26, 4, 2]) to construct the semiclassical approximation of the resolvent $(h^2 H - z)^{-1}$ which has the following form:

$$(h^2 H - z)^{-1} = \sum_{0 \leq j \leq N-1} h^j q_j^h(z, x, hD) + h^N r_N^h(z, x, hD)(h^2 H - z)^{-1}, \quad (\text{A.4})$$

where $q_j^h \in S(\langle x \rangle^{-j} \langle \xi \rangle^{-j}, g)$ are of the forms

$$q_0^h(z, x, \xi) = \frac{1}{p^h(x, \xi) - z}, \quad q_j^h(z, x, \xi) = \sum_{0 \leq k \leq j} \frac{q_{jk}^h(x, \xi)}{(p^h(x, \xi) - z)^{1+j+k}}, \quad j \geq 1,$$

with $q_{jk}^h \in S(\langle x \rangle^{-j} \langle p^h(x, \xi) \rangle^{N_j(k)}, g)$ with some integer $N_j(k)$. Moreover, the remainder r_N^h belongs to $S(\langle x \rangle^{-N} \langle \xi \rangle^{-N}, g)$ with the bounds

$$|\partial_x^\alpha \partial_\xi^\beta r_N^h(z, x, \xi)| \leq C_{N\alpha\beta} \langle x \rangle^{-N-|\alpha|} \langle \xi \rangle^{-N-|\beta|} |\operatorname{Im} z|^{-2N-1-|\alpha+\beta|},$$

uniformly in $x, \xi \in \mathbb{R}^d$ and $h \in (0, 1]$, and locally uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$. In particular, if $N > d$ then $r_N^h(z, x, hD)$ is bounded from L^2 to L^q with the bounds

$$\|r_N^h(z, x, hD)\|_{L^2 \rightarrow L^q} \lesssim h^{-d(1/2-1/q)} |\operatorname{Im} z|^{-n(N,q)}, \quad q \in [2, \infty], \quad h \in (0, 1], \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (\text{A.5})$$

where $n(N, q)$ is a positive number depending on N and q .

We now plug the approximation (A.4) into the well-known Helffer-Sjöstrand formula [15]:

$$f(h^2 H) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z) (h^2 H - z)^{-1} dz \wedge d\bar{z},$$

where $dz \wedge d\bar{z} = -2idudv$ with $z = u + iv$ and \tilde{f} is an almost analytic extension of f . Note that, since $f \in C_0^\infty(\mathbb{R})$, \tilde{f} is also compactly supported and, for any $M \geq 0$,

$$|\partial_{\bar{z}} \tilde{f}(z)| \leq C_M |\operatorname{Im} z|^M \quad (\text{A.6})$$

with some $C_M > 0$. Let us set the symbol χ^h defined by

$$\chi^h = \sum_{j=0}^{N-1} h^j \chi_j^h \quad \text{with} \quad \chi_0^h = f \circ p^h, \quad \chi_j^h = \sum_{k=0}^j \frac{(-1)^{j+k}}{(j+k)!} q_{jk}^h \cdot f^{(j+k)} \circ p^h, \quad j = 1, 2, \dots, N-1.$$

Then, $\{\chi^h\}_{h \in (0,1]}$ is bounded in $S(1, g)$ and $\operatorname{supp} \chi^h \subset \operatorname{supp} f \circ p^h$. Moreover, taking $N > d$ we learn by (A.5) and (A.6) that the remainder

$$R_N^h := f(h^2 H) - \chi^h(x, hD) = -\frac{h^N}{2\pi i} \int_{\operatorname{supp} \tilde{f}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z) r_N^h(z, x, D) (h^2 H - z)^{-1} dz \wedge d\bar{z}$$

satisfies

$$\|R_N^h\|_{L^2 \rightarrow L^q} \leq C_{Nq} h^{N-d(1/2-1/q)} \int_{\operatorname{supp} \tilde{f}} |\operatorname{Im} z|^{M-n(N,q)-1} dz \wedge d\bar{z} \leq C'_{Nq} h^{N-d(1/2-1/q)},$$

provided that $M \geq n(N, q) + 1$, which complete the proof for the high energy part.

The low energy part is also verified by the same argument as above with $h = 1$. \square

Remark A.3. Assume (for simplicity) that $A \equiv 0$. It is easy to see that $[x, H]$ is bounded in x . Since H is elliptic, we then learn by a standard commutator argument (see, *e.g.*, [2, Section 2]) that there exists $N > 0$ such that $(H - z)^{-N}$ is bounded on L^q for any $q \in [1, \infty]$ with norm dominated by a power of $\langle z \rangle |\operatorname{Im} z|^{-1}$. Therefore, using the same argument as that in [2, Section 2] one can prove in this case that $f(h^2 H)$ is bounded from $L^{q'}$ to L^q for any $1 \leq q' \leq q \leq \infty$ with the norm of order $h^{-d(1/q'-1/q)}$. However, since the $L^2 \rightarrow L^q$ boundedness is sufficient to study *local-in-time* Strichartz estimates, we do not write here the precise statement.

Proof of Corollary 1.5. Let f be as in Lemma A.1 and choose $F \in C_0^\infty(\mathbb{R})$ so that $F \equiv 1$ on $\text{supp } f$ and $\text{supp } F \Subset (0, \infty)$. We learn by Proposition A.2 (1), Theorem 4.3 and the TT^* -argument that

$$\|F(h^2 H)e^{-itH}u_0\|_{L_T^p L^q} \leq C_T h^{(1-2/m)/p} \|u_0\|_{L^2}$$

for any admissible pair (p, q) . Since $f(h^2 H) = f(h^2 H)F(h^2 H)$ by the spectral decomposition theorem, the estimate (A.2), Proposition A.2 (2) and the above estimates imply that

$$\begin{aligned} \|e^{-itH}u_0\|_{L_T^p L^q} &\leq C_T \|u_0\|_{L^2} + C \left(\sum_h \|e^{-itH}F(h^2 H)f(h^2 H)u_0\|_{L_T^p L^q}^2 \right)^{1/2} \\ &\leq C_T \|u_0\|_{L^2} + C_T \left(\sum_h h^{2(1-2/m)/p} \|f(h^2 H)u_0\|_{L^2}^2 \right)^{1/2} \\ &\leq C_T \|\langle H \rangle^{(1/2-1/m)/p} u_0\|_{L^2} \end{aligned}$$

provided that (p, q) is admissible and $q < \infty$. □

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